Geometric and Algebraic Topology
Comprehensive Exam – January 2014

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Notation: $\mathbb{R}$ is the set of real numbers. B^2 is the closed unit disk in $\mathbb{R}^2$ with center $(0,0)$. S^1 is the unit circle in $\mathbb{R}^2$ with center $(0,0)$. $\mathbb{Z}$ is the infinite cyclic group. $\mathbb{Z}_m$ denotes $\mathbb{Z}/m\mathbb{Z}$.

Geometric Topology

1. List all the compact, connected surfaces M with $\chi(M) \geq 0$. Include surfaces which are orientable, non-orientable, with boundary, and without boundary.

2. Find the fundamental groups of the following spaces. Give very brief reasons. Hint: Draw pictures.

   (a) $(B^2 \times S^1) - (\{(0,0)\} \times S^1)$

   (b) $\mathbb{R}^2-$ two points.

   (c) $\mathbb{R}^3-$ two points.

   (d) $\mathbb{R}^3 - (A \cup C_0 \cup C_1)$, where A is the z axis, $C_0 = S^1 \times \{0\}$, and $C_1 = S^1 \times \{1\}$.

3. Suppose $h : B^2 \to B^2$ is a homeomorphism. Prove that $h(S^1) = S^1$. Hints: Suppose x is a point in S^1 such that $h(x) \notin S^1$. Consider the restriction g of h to $B^2 - \{x\}$. Think about π_1 or H_1; you may quote standard facts about these functors and their values on standard spaces. If you need a specific homeomorphism or homotopy you do not need to give a formula for it; just say very briefly and informally what properties you need it to have.

Algebraic Topology ($H_p(X)$ denotes homology with $\mathbb{Z}$ coefficients $H_p(X;\mathbb{Z})$.)

4. X is a space with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_{10}$, $H_2(X) = \mathbb{Z}_3$, and all other homology groups zero. Compute all the homology groups $H_p(X;\mathbb{Z}_2)$ of X with $\mathbb{Z}_2$ coefficients.

5. Recall that the suspension ΣX of a space X is the quotient space obtained from $X \times [-1,1]$ by shrinking the subspace $X \times \{1\}$ to a point and the subspace $X \times \{-1\}$ to a (different) point. For example, if X is the two point space $\{a,b\}$, then ΣX is a quadrilateral. State and prove a formula which expresses the homology groups (with $\mathbb{Z}$ coefficients) of ΣX in terms of those of X. Hint: $\Sigma X = C^+X \cup C^-X$, where the cone C^+X is the image of $X \times [0,1]$ in ΣX and the cone C^-X is the image of $X \times [-1,0]$ in ΣX.

THERE IS ONE MORE PROBLEM ON THE NEXT PAGE

6. Let $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$ be chain complexes.
Let $\phi = \{\phi_p : C_p \to C'_p\}$ and $\psi = \{\psi_p : C_p \to C'_p\}$ be chain maps from $\mathcal{C}$ to $\mathcal{C}'$.
Let $\Delta = \{\Delta_p : C_p \to C'_{p+1}\}$ be a chain homotopy between ϕ and ψ.

(a) Define what it means for $\mathcal{C}$ to be a chain complex.

(b) Define what it means for ϕ to be a chain map.

(c) Define what it means for Δ to be a chain homotopy between ϕ and ψ.

(d) Prove that if ϕ and ψ are chain homotopic as above, then they induce the same homomorphisms $(\phi_*)_p = (\psi_*)_p$ from $H_p(\mathcal{C})$ to $H_p(\mathcal{C}')$.