Geometric and Algebraic Topology
Comprehensive Exam – August 2013

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise give clear and complete proofs of your claims.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

1. Let M be the surface obtained by identifying the edges of an octagon using the pattern $abcdba^{-1}d^{-1}c^{-1}$. Compute the Euler characteristic of M and identify which surface it is.

2. Let S^1 be the circle, and let $p \in S^1$. Let $X = S^1 \times S^1$ and $A = \{p\} \times S^1$.
   (a) Is A a retract of X? If yes, give an explicit retraction map r. If no, give a proof that it is not.
   (b) Is A a deformation retract of X? If yes, give an explicit deformation retraction map f. If no, give a proof that it is not.

3. Let X be a path connected, locally path connected space. Let $x_0 \in X$. Assume that $\pi_1(X, x_0)$ is finite. Prove that every continuous map $f : X \to S^1$ is homotopic to a constant map, where S^1 is the circle. Hint: Consider the universal covering space of S^1.

Algebraic Topology (All homology and cohomology groups have $\mathbb{Z}$ coefficients.)

4. Let K be the Klein bottle obtained by identifying the edges of a square using the pattern $aba^{-1}b$. You are given that $H_2(K) = 0$ and $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$, where the $\mathbb{Z}$ summand is generated by the homology class of the cycle a and the $\mathbb{Z}_2$ summand is generated by the homology class of the cycle b. Use the Mayer-Vietoris sequence to compute the first and second homology groups of the following spaces.
   (a) X is obtained by attaching a disk D to K so that ∂D is identified with a.
   (b) Y is obtained by attaching a disk D to K so that ∂D is identified with b.

5. X is a space with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_6$, $H_2(X) = \mathbb{Z}_4$, and all other homology groups zero. Compute all the cohomology groups of X.

6. X is a space with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_4$, $H_2(X) = \mathbb{Z}_9$ and all other homology groups zero. Y is a space with $H_0(Y) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_6$ and all other homology groups zero. Compute all the homology groups of $X \times Y$. Y