Geometric and Algebraic Topology
Comprehensive Exam – May 2013

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; unless stated otherwise give clear and complete proofs of your claims.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

1. Let M_g denote the connected sum of g tori; let N_g denote the connected sum of g projective planes. Classify each of the following surfaces as an M_g or N_g, specifying g.

   (a) The connected sum of a torus and a Klein bottle.

   (b) The compact orientable surface S with Euler characteristic $\chi(S) = -6$.

   (c) The compact non-orientable surface S with Euler characteristic $\chi(S) = -4$.

   (d) The surface represented by a hexagon with the edges identified according to the symbol $abcb^{-1}c^{-1}a^{-1}$.

2. Give presentations for the fundamental groups of the following spaces. P denotes a projective plane.

   (a) $P \times P$

   (b) $P\#P$, where "#" denotes connected sum.

   (c) $P \vee P$, where "$\vee$" denotes the gluing of the two spaces together at a single point.

   (d) $P - \{x\}$, where $x \in P$.

3. Let $p : \widetilde{X} \to X$ be a covering space. $\widetilde{X}$ and X are assumed to be path connected and locally path connected. Let $\tilde{x}_0 \in \widetilde{X}$. Let $x_0 = p(\tilde{x}_0)$.

   (a) Prove that $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is one-to-one.

   (b) Let $\tilde{x}_1 \in p^{-1}(x_0)$. Prove that $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ and $p_*(\pi_1(\widetilde{X}, \tilde{x}_1))$ are conjugate subgroups of $\pi_1(X, x_0)$.

CONTINUED ON THE NEXT PAGE

Algebraic Topology (All homology and cohomology groups have $\mathbb{Z}$ coefficients.)

4. (a) State the Poincaré Duality Theorem.

   (b) State the Universal Coefficient Theorem for cohomology (in terms of homology groups).

   (c) Prove that every closed, orientable 3-manifold M has Euler characteristic $\chi(M) = 0$.

5. $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$,

   $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, z = 0\}$,

   $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, y = 0, -1 \leq z \leq 1\}$

   Find the homology groups $H_1(X)$ and $H_2(X)$ for each of the following spaces. No proofs are required.

   (a) $X = S \cup D$

   (b) $X = S \cup A$

   (c) $X = S \cup D \cup A$

6. Let S^2 be the 2-sphere and S^1 the circle, where we regard S^1 as the equator of S^2. Compute the relative homology groups $H_2(S^2, S^1)$ and $H_1(S^2, S^1)$. You may assume that the groups $H_p(S^n)$ are what they are.