Geometric and Algebraic Topology

Comprehensive exam — January 2012

Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims.

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Geometric Topology

1. Let S denote the 2-sphere and X a topological space. Let $f : X \to S$ be a continuous map that is not onto. Prove that f is homotopic to a constant map.

2. Let $p : Y \to X$ be a covering map and A a connected topological space. Suppose f and g are continuous maps from A to Y such that $p \circ f = p \circ g$. Suppose further that there is a point a_0 in A such that $f(a_0) = g(a_0)$. Prove $f(a) = g(a)$ for each a in A.

3. Let F be an orientable 2-manifold with Euler characteristic χ and J a simple closed curve in F that does not separate F. Construct G from F by performing surgery on J. That is, we get G by deleting the interior of a regular neighborhood of J from F and then gluing disks onto the two resulting boundary components.

   (a) Calculate the Euler characteristic of G.

   (b) If J' is a second non-separating simple closed curve on F, and if G' is obtained by doing surgery on J', prove that G is homeomorphic to G'.

Algebraic Topology

1. Let M denote the Möbius band and M' its boundary circle. It is a fact that the inclusion induced map $i_* : H_1(M') \to H_1(M)$ is given by $i_*(x) = 2x$.

   (a) Calculate the homology groups of the pair (M, M').

   (b) Let $2M$ denote the *double* of M. That is, $2M$ is obtained by gluing two copies M_1 and M_2 of M together along their common circle boundaries. (Hence $2M = M_1 \cup M_2$, and $M_1 \cap M_2$ is the common boundary circle.) Calculate the homology groups of $2M$.

2. Let S denote a k dimensional sphere and T an n dimensional sphere. We assume $k \neq n$ and both are at least 2. Calculate the homology groups of $S \times T$.

3. Let X be a connected, closed, orientable $\cancel{4}$ 5-manifold with $H_1(X) = Z_6$ and $H_2(X) = Z \oplus Z_{15}$. Compute all other homology and cohomology groups of X.