Geometric and Algebraic Topology

Comprehensive Exam — August 2009

Partial credit will be given for your work. Credit equivalent to five complete problems will guarantee a pass.

Algebraic topology

1. Let M be a closed connected orientable n-manifold. Show that $H_{n-1}(M;\mathbb{Z})$ is free abelian. Give an example of a closed orientable manifold with torsion in homology, and verify your example.

2. Let $X = X_1 \cup X_2$, where X_1 and X_2 are both tori $S^1 \times S^1$, and $X_1 \cap X_2$ is a simple closed curve which bounds a disc in X_1 and is essential in X_2. Use the Mayer-Vietoris sequence to find the homology of X.

3. Let S_g be the closed orientable surface of genus g, and let $f : S_2 \to S_3$ be a continuous map. Write down the ring structure on the cohomology of S_2 and S_3; no proof is required. Show that $f^* : H^2(S_3) \to H^2(S_2)$ must be zero, by any method.

Geometric topology

4. Let M be the surface obtained by identifying the edges of an octagon using the pattern $abcbdc^{-1}da^{-1}$. Compute the Euler characteristic of M, and identify which surface it is.

5. Find the fundamental group of the space produced by attaching a disc to a torus, namely by gluing the boundary of a disc to an essential simple closed curve on the torus by a homeomorphism.

   Show that the resulting space is not homotopy equivalent to a circle.

6. Find all two-fold covers of $S^1 \vee S^1$ up to covering space isomorphism, carefully justifying your answer. What does this tell you about the rank of index two subgroups of the free group on two generators?