Real Analysis Comprehensive Exam
Date: August 2018

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Define μ on the subsets of $\mathbb{N}$ as follows: If E is empty, define $\mu(E) = 0$; if E is finite and nonempty, define $\mu(E) = \sum_{n\in E} 1/2^n$; if E is infinite, define $\mu(E) = \infty$.

(a) Find an increasing sequence $\{E_j\}$ of subsets of $\mathbb{N}$ so that

$$\lim_{j\to\infty} \mu(E_j) \neq \mu\left(\bigcup_{j=1}^{\infty} E_j\right).$$

(b) Determine whether μ is a measure.

(2) In parts (a) and (b), indicate whether the given statement is true or false. Briefly explain your answer for those you mark as true, and give a relevant example for those you mark as false. In each case $\{f_n\}$ denotes a sequence of Lebesgue measurable functions.

(a) If $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0, 1]$ then $f_n \to 0$ in measure (with respect to Lebesgue measure on $[0, 1]$).

(b) If $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in \mathbb{R}$ then $f_n \to 0$ in measure (with respect to Lebesgue measure on $\mathbb{R}$).

(c) For $n \geq 1$ define $f_n = n\chi_{(0,1/n)}$. Here χ_E denotes the characteristic (or indicator) function of the set E. Show that (using Lebesgue measure) $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \neq \int_{\mathbb{R}} \lim_{n\to\infty} f_n$, and explain why this example does not contradict the Dominated Convergence Theorem.

(3) Suppose that μ and ν are finite measures on a measurable space $(X, \mathcal{M})$, ν is absolutely continuous with respect to μ, and f is a nonnegative measurable function on $(X, \mathcal{M})$. Show that

$$\int f \frac{d\nu}{d\mu} d\mu = \int f d\nu.$$

Here $\dfrac{d\nu}{d\mu}$ denotes the Radon-Nikodym derivative of ν with respect to μ.

(4) Suppose $f \colon [a, b] \to \mathbb{R}$. (Here $-\infty < a < b < \infty$.)

(a) Define what it means for f to be absolutely continuous on $[a, b]$.

(b) Suppose that f is continuous on $[a, b]$. Assume that the derivative f' exists a.e. and is integrable with respect to Lebesgue measure. Is f absolutely continuous? Give a brief explanation.

(c) Is it possible for f to be increasing on $[a, b]$ but not differentiable at any point? Give a brief explanation.

(5) Let S denote the collection of functions of the form

$$a_0 + a_1 \cos x + \cdots + a_n \cos nx$$

with n a nonnegative integer and $a_0, \ldots, a_n$ real. For which functions f on the interval $[0, \pi]$ is there a sequence in S converging uniformly to f on $[0, \pi]$? Be sure to give a proof.

(6) Let X and Y be Banach spaces. Let $\{T_n\}$ be a sequence of bounded linear transformations from X to Y such that $\lim T_n x$ exists for each $x \in X$. Define $T \colon X \to Y$ by $Tx = \lim T_n x$. Assuming that T is a linear transformation, show that T is bounded.