Real Analysis Comprehensive Exam
Date: June 2018

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Prove or disprove: There is a Borel probability measure μ on $\mathbb{R}$ so that (a) $\mu(\{p\}) = 0$, for all $p \in \mathbb{R}$, and (b) $\mu(E) = 0$ or 1, for all Borel sets $E \subseteq \mathbb{R}$. Recall that a probability measure on $\mathbb{R}$ is a measure so that $\mu(\mathbb{R}) = 1$.

(2) Parts (a) and (b) state properties of a sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative Lebesgue measurable functions on $[0,1]$. In each part, either give an example with the given property or prove that no such sequence exists.
  (a) $\lim_{n\to\infty} f_n(x) = 0$ a.e. and $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \infty$
  (b) $\lim_{n\to\infty} f_n(x) = \infty$ a.e. and $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0$

(3) Do the following.
  (a) Show that $L^2([0,1]) \subseteq L^1([0,1])$. Also, give an example illustrating that equality does not hold.
  (b) Prove or disprove: $L^1([1,\infty)) \subseteq L^2([1,\infty))$.
  (c) Give an example of a nonconstant differentiable function that is in $L^1(\mathbb{R})$. Verify your answer.

(4) Suppose X and Y are normed vector spaces.
  (a) Give the definition of a linear function from X to Y.
  Give the definition of a continuous function from X to Y.
  Describe the open sets in X.
  (b) Prove that if a linear function $T : X \to Y$ is continuous, then it is bounded. [Recall that a linear function $T : X \to Y$ is called *bounded* if there is a constant C so that $\|T(x)\| \le C\|x\|$, for all $x \in X$.]

(5) In this problem we use Lebesgue measure. For a given $g \in L^1[0,1]$ define ϕ_g on $C[0,1]$ by $\phi_g(f) = \int_0^1 f(x)g(x)\, dx$.
  (a) Show that ϕ_g is a bounded linear functional on $C[0,1]$.
  (b) Define a bounded linear functional ϕ on $C[0,1]$ by $\phi(f) = f(1)$. Prove that there is no $g \in L^1[0,1]$ so that $\phi = \phi_g$.

(6) Let B denote the set of continuous functions f on $[0,1]$ so that $|f(x)| \le 1$ for all $x \in [0,1]$.
  (a) Is B a compact subset of $C[0,1]$? Explain.
  (b) For $f \in B$ define $I[f]$ on $[0,1]$ by $I[f](x) = \displaystyle\int_0^x f(t)dt$. Prove that if $\{f_n\}$ is a sequence in B then it has a subsequence $\{f_{n_k}\}$ so that $\{I[f_{n_k}]\}$ converges uniformly on $[0,1]$.