Real Analysis Comprehensive Exam
January 2018

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Consider $\mathbb{Z} \subset \mathbb{R}$. Define
$$\mathcal{A} := \{A \subset \mathbb{R} : A \subset \mathbb{Z} \text{ or } A^c \subset \mathbb{Z}\}.$$
  (a) Show that $\mathcal{A}$ is a σ-algebra on $\mathbb{R}$.
  (b) Prove or disprove: (i) $\mathcal{B}_{\mathbb{R}} \subset \mathcal{A}$. (ii) $\mathcal{A} \subset \mathcal{B}_{\mathbb{R}}$.
  ($\mathcal{B}_{\mathbb{R}}$ denotes the σ-algebra of Borel sets in $\mathbb{R}$.)

(2) Compute
$$\lim_{n\to\infty} \int_0^n \frac{x^n}{1+x^n}\, e^{-2x}\, dx.$$
Justify your computations.

(3) Consider ℓ^∞, the bounded sequences of complex numbers. One may define a norm on this space by
$$||f|| = \sup_n \left\{ \frac{|f(n)|}{n} \right\}.$$
(Note that this is not the usual sup norm.) Prove or disprove that ℓ^∞ is complete with respect to this norm.

(4) Give proof or counter example for each of the following.
  (a) $L^p(\mathbb{R}) \subset L^q(\mathbb{R})$, for all $1 \le p < q < \infty$.
  (b) $L^q(\mathbb{R}) \subset L^p(\mathbb{R})$, for all $1 \le p < q < \infty$.
  (c) $L^q([0,1]) \subset L^p([0,1])$, for all $1 \le p < q < \infty$.

(5) If $f \in L^\infty([0,1])$, then
$$\lim_{n\to\infty} \left(\int_0^1 |f(x)|^n dx \right)^{\frac{1}{n}} = ||f||_\infty.$$

(6) Consider $\mathbb{R}$ with Lebesgue measure m. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}$ converging almost everywhere to f. Prove that there exists a sequence $\{E_k\}$ of measurable sets so that $f_n \to f$ uniformly on each set E_k and $m(\mathbb{R} \setminus \cup_k E_k) = 0$. (Hint: Use Egoroff's theorem; it might be useful to write $\mathbb{R} = \cup_k (-k, k)$.)