Real Analysis Comprehensive Exam
May 2017

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Lebesgue measure is denoted by m. Complete solutions to four of the six problems will guarantee a pass.

(1) Let X be a set and S a countable infinite subset of X. Define

$$\mathcal{A} := \{A \subset X : A \subset S \text{ or } A^c \subset S\}.$$

(a) Show that $\mathcal{A}$ is a σ-algebra.
(b) Is $\mathcal{A}$ countable or uncountable? Explain your reasoning in a couple of sentences.

(2) Let $D = \{(x, y) : -1 \leq x, y \leq 1\}$. Define

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Is f an L^1 function on D (with respect to Lebesgue measure)? Justify your answer.

(3) Consider ℓ^2, the space of sequences of complex numbers (a_n) so that $\sum |a_n|^2 < \infty$. Let λ_n be a sequence of positive real numbers so that $\sum \lambda_n < \infty$. Set

$$\langle a\,, b\rangle_\lambda = \sum \lambda_n a_n \overline{b_n}, \quad a, b \in \ell_2.$$

(a) Show that $\langle\ ,\ \rangle_\lambda$ is a well-defined inner product on ℓ^2.
(b) Show that ℓ^2 is not complete with respect to the norm $||a||_\lambda := (\langle a\,, a\rangle_\lambda)^{1/2}$.

(4) Suppose that $f_n \to f$ in $L^2([0,1], m)$, as $n \to \infty$. Determine whether the following statements are correct (give a proof if the answer is positive and a counterexample if negative):
(a) $f_n \to f$ in $L^1([0,1], m)$.
(b) $f_n \to f$ a.e. on $[0,1]$.
(c) $f_n \to f$ weakly in $L^2([0,1], m)$.

(5) Assume f and $f_n, n = 1, 2 \ldots$, are in $L^1(\mathbb{R}, m)$ and $\displaystyle\sum_1^\infty \int_{\mathbb{R}} |f - f_n| dm < \infty$. Show that f_n converges to f almost everywhere. (Be sure to state any theorem you use.)

(6) Suppose that $f : [0,1] \to \mathbb{R}$ is absolutely continuous. Define

$$g(x) := \int_0^1 f(xy)dy.$$

Show that $g : [0,1] \to \mathbb{R}$ is absolutely continuous.