Real Analysis Comprehensive Exam
Date: January 2017

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Which of the following functions is in $L^1((0,\infty))$?
(a) $\dfrac{\sin(x)}{x}$, (b) $\dfrac{\sin(x)}{\sqrt{x}}$, (c) $\dfrac{\sin(x)}{\sqrt{x^3}}$.
In each case give a proof for your answer.

(2) Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of measurable functions converging pointwise to f. Assume that the sequence of real numbers $\{\, \|f_n\|_2 \,\}$ is bounded.
(a) Show that $f \in L^1(X, \mu)$.
(b) Prove that $f_n \to f$ in L^1.
(Hint: You may want to use Fatou's Lemma and/or Egoroff's Theorem.)

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space.
(a) Give the definition of $L^\infty(X, \mu)$.
(b) Show that if a measurable function f is not in $L^\infty(X, \mu)$, then there exists a sequence $a_n \in \mathbb{R}$ and a sequence of disjoint measurable sets $\{X_n\}$ so that for all $n \in \mathbb{N}$ (i) $a_n > n$, (ii) $\mu(X_n) > 0$, and (iii) $|f(x)| > a_n$, for $x \in X_n$

(4) Asume (X, μ) and (Y, λ) are σ-finite measure spaces. Let $\{f_i\}$ be a countable orthonormal basis of $L^2(X, \mu)$ and let $\{g_j\}$ be a countable orthonormal basis of $L^2(Y, \lambda)$. Show that if $h_{ij}(x, y) := f_i(x)g_j(y)$, then $\{h_{ij}\}$ is an orthonormal basis of $L^2(X \times Y, \mu \times \lambda)$. (You may assume that h_{ij} is measurable.)

(5) Recall that a sequence x_n in Banach space X converges weakly to x means that $\lambda(x_n) \to \lambda(x)$ for every $\lambda \in X^*$. In this problem we consider $X = \ell^1(\mathbb{N})$.
(a) Describe $(\ell^1(\mathbb{N}))^*$. (No justification is necessary, but be explicit.)
(b) Show that a sequence f_n converges weakly in $\ell^1(\mathbb{N})$ implies that f_n converges pointwise.
(c) Give an example of a sequence f_n in $\ell^1(\mathbb{N})$ so that (i) f_n converges pointwise to 0 and (ii) $\|f_n\|_1 = 1$ (all n), but f_n does not converge weakly to 0.

(6) Prove that if $f \in L^1(\mathbb{R})$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} \cos(nx) f(x)\, dx = 0.$$

(Hint: Consider characteristic (indicator) functions χ_E; first $\chi_{(a,b)}$, then χ_U with U open of finite measure, then χ_E with E of finite measure.)