Real Analysis Comprehensive Exam
Date: August 2016

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Prove that

$$\lim_{n\to\infty}\int_0^1 \sin\left(\frac{x}{n}\right)\frac{n^3}{1+n^2x}dx = 1.$$

(2) Let m denote Lebesgue measure on $\mathbb{R}$, and fix $f \in L^1(\mathbf{R}, m)$.

(a) Define the sequence $\{f_n\}$ on $\mathbf{R}$ by $f_n(x) = f(x-n)$. Show that

$$\int_{[0,1]}\sum_{n=1}^{\infty}|f_n|\, dm < \infty.$$

(b) Show that $\lim_{n\to\infty} f(x-n) = 0$ for a.e. x.

(3) (a) State appropriate assumptions on a measure space $(X, \mathcal{M}, \mu)$ that guarantee the following statement to hold.

($*$) "If $\{f_n\}$ is a sequence of measurable functions on X with values in $\mathbb{R}$ so that f_n converges to a measurable function f almost everywhere, then f_n converges to f in μ-measure."

(b) Prove that ($*$) holds under your assumptions.

(c) Show that if your assumptions fail, then ($*$) may be false.

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that $f \in \cap_{1\le p<\infty} L^p(X, \mu)$. For $1 \le p < \infty$ define

$$\phi(p) = \log ||f||_p^p.$$

Prove that, for every $p, q \ge 1$ and for every pair (α, β) of positive numbers with $\alpha + \beta = 1$, we have $\phi(p\alpha + q\beta) \le \alpha\, \phi(p) + \beta\phi(q)$.

(5) Let X be a Hilbert space. Let $T\colon X \to X$ be a linear operator which is *symmetric*, in the sense that

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

for all $x, y \in X$. Show that T is bounded.
(Hint: If $x_n \to x$ and $Tx_n \to z$ in X, for fixed y in X evaluate $\lim \langle Tx_n, y\rangle$ in two ways.)

(6) Let $\ell_1 = \{(a_n)\colon a_n \in \mathbf{C}, \sum |a_n| < \infty\}$ and $c_0 = \{(b_n)\colon b_n \in \mathbf{C},\ \lim b_n = 0\}$, and for $(b_n) \in c_0$ define $||(b_n)|| = \sup |b_n|$.

(a) Let $a = (a_n) \in \ell_1$. Prove that the map $T_a\colon c_0 \to \mathbf{C}$ defined by $(b_n) \mapsto \sum a_n b_n$ is a bounded linear functional on c_0. (Include detailed arguments.)

(b) Show that if $T\colon c_0 \to \mathbf{C}$ is a bounded linear functional, then there exists $a \in \ell_1$ such that $T = T_a$. (Hint: You may use the fact that the subspace of c_0 consisting of sequences with only a finite number of nonzero entries is dense in c_0.)