Real Analysis Comprehensive Exam
Date: June 2016

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Let μ be a finite measure on $(X, \mathcal{M})$. Assume that $\mu(E) > 0$ if $E \in \mathcal{M}$ and E is nonempty. For $x \in X$ define

$$\lambda(x) = \inf\{\mu(E) \colon E \in \mathcal{M}, x \in E\}.$$

(a) Show that for each $x \in X$ there exists $A_x \in \mathcal{M}$ so that $x \in A_x$ and $\mu(A_x) = \lambda(x)$.

(b) We use the notation from part (a). Fix $x, y \in X$ so that A_x and A_y are not disjoint. Show that $x \in A_y$. (Suggestion: First show that $\mu(A_x) > \mu(A_x \setminus A_y)$.)

(2) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_0^\infty$ be a sequence in $L^1(X, \mu)$. Assume that $\{f_n\}$ converges almost everywhere to a function $f \in L^1(X, \mu)$. Furthermore, assume that for all n

$$||f_n||_1 \leq ||f||_1.$$

Prove that $\lim_{n\to\infty} \int_X f_n(x) d\mu(x) = \int_X f(x) d\mu(x)$. (Hint: At some point you might want to consider the function $|f| + |f_n| - |f - f_n|$.)

(3) Let $1 < p, q, s < \infty$ so that $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$. Assume that $f \in L^p(\mathbb{R}^n, m)$, $g \in L^q(\mathbb{R}^n, m)$, and $h \in L^s(\mathbb{R}^n, m)$. (Here m denotes Lebesgue measure on $\mathbb{R}^n$.) Prove that $fgh \in L^1(\mathbb{R}^n, m)$.

(4) Let m denote Lebesgue measure on the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. If f is a nonnegative measurable function on X, let

$$\mathcal{R} = \{(x, y) \in X \times [0, \infty) : 0 \leq y \leq f(x)\}.$$

Prove that the measure of $\mathcal{R}$ with respect to $\mu \times m$ is $\int_X f d\mu$. You may assume that $\mathcal{R}$ is an $\mathcal{M} \otimes \mathcal{B}$ measurable subset of $X \times \mathbb{R}$.

(5) Let f be an absolutely continuous function on $[a, b] \subset \mathbb{R}$. Assume that f is never zero on $[a, b]$. Prove that $\frac{1}{f}$ is absolutely continuous on $[a, b]$.

(6) If X is a Banach space with norm $||\cdot||$, we say that X has property $(*)$ if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $x, y \in X$ with $||x|| = ||y|| = 1$ and

$$\left|\left|\frac{x+y}{2}\right|\right| \geq 1 - \delta$$

then $||x - y|| < \epsilon$.

(a) Prove that every Hilbert space has property $(*)$. (Hint: The parallelogram law may be useful.)

(b) Does ℓ^∞ have property $(*)$? (Here ℓ^∞ equals $L^\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where μ is counting measure on the σ-algebra $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$.)