Real Analysis Comprehensive Exam
Date: August 2015

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Let $(X, \mu, \mathcal{M})$ be a measure space. Consider a sequence of measurable sets $\{E_n\}$ so that $\lim_{n\to\infty} \mu(X \setminus E_n) = 0$. Show that the set of $x \in X$ which are in only finitely many E_n has measure 0.

(2) Let $(X, \mathcal{B}, \mu)$ be a finite measure space. Consider the following three types of convergence.
(a) Convergence almost everywhere.
(b) Convergence in measure.
(c) Convergence in L^1-norm.
For each of the six implications, (a) $\Longrightarrow$ (b), (a) $\Longrightarrow$ (c), (b) $\Longrightarrow$ (a), (b) $\Longrightarrow$ (c), (c) $\Longrightarrow$ (b), and (c) $\Longrightarrow$ (a), provide a proof or counterexample.

(3) Prove or disprove that $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{1}{\sqrt[3]{n^2}} \chi_{[n,2n]}$ converges weakly to 0 in $L^2(\mathbb{R})$.

(4) Prove that the product of two absolutely continuous functions on a finite interval $[a, b]$ is absolutely continuous on $[a, b]$.

(5) Let $X = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Prove that

$$||f|| := ||f||_1 + ||f||_2$$

defines a norm on X and that X is complete with respect to this norm.

(6) Use Fubini-Tonelli and the identity

$$\frac{1}{x} = \int_0^\infty e^{-xt}\, dt$$

to prove the value of the improper integral

$$\int_0^\infty \frac{\sin(x)}{x}\, dx$$

is $\frac{\pi}{2}$. Justify each step in your calculation.