Real Analysis Comprehensive Exam
Date: June 2015

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Lebesgue measure on $\mathbb{R}$ is denoted by m. Complete solutions to four of the six problems will guarantee a pass.

(1) Compute the limit

$$\lim_{n\to\infty} \int_0^1 \frac{t^{-1/n}}{(1+t/n)^n} dt.$$

Give full justification.

(2) Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let $\{E_n : n \in \mathbb{N}\}$ be a family of measurable sets so that $\sum_n \mu(E_n) < \infty$. Prove that the set $E := \{x : x \in E_n \text{ for infinitely many } n\}$ is measurable and has measure zero.

(3) Consider a measure space $(X, \mathcal{M}, \mu)$ and a sequence of real-valued measurable functions (f_n) on X.

(a) Give the definition of '$f_n \to f$ in measure'.

(b) Prove or disprove: if f_n is a sequence in $L^2(\mathbb{R})$ so that for each $\varphi \in L^2(\mathbb{R})$ $\int_{\mathbb{R}} f_n \varphi \, dx \to 0$, then $f_n \to 0$ in measure.

(c) Prove or disprove: if f_n is a sequence in $L^2([0,1])$ so that for each $\varphi \in L^2([0,1])$ $\int_{[0,1]} f_n \varphi \, dx \to 0$, then $f_n \to 0$ in measure.

(4) Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and let $1 < p < \infty$. It follows from the Hölder inequality that if $\mu(X) < \infty$, then $L^p(X, \mu) \subset L^1(X, \mu)$. Prove the converse: if $L^p(X, \mu) \subset L^1(X, \mu)$, then $\mu(X) < \infty$.

(5) For $1 \le p < \infty$ and $f \in L^p(\mathbb{R})$, prove

$$\lim_{a\to 0} \int |f(x+a) - f(x)|^p \, dm(x) = 0.$$

(Hint: Do this for $f \in C_c(\mathbb{R})$ first.)

positive

(6) Let A, B be two Lebesgue measurable sets in $\mathbb{R}$ of finite measure. Define

$$h(x) := \int_{\mathbb{R}} \chi_A(y - x)\chi_B(y) \, dm(y).$$

(a) Compute $h(x)$.

(b) Show that $m((x + A) \cap B) \neq 0$ for some $x \in \mathbb{R}$.

(Notation: $\chi_E : \mathbb{R} \to \mathbb{R}$ is the 'indicator' function defined to be 1 on E and 0 elsewhere. $x + A := \{x + a : a \in A\}$.)