Real Analysis Comprehensive Exam
Date: January 2015

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Consider the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \sum_n \frac{1}{1 + n^4 x^2}.$$

Prove the following
  (a) f is continuous on $(0, \infty)$.
  (b) f is differentiable on $(0, \infty)$.
  (c) $f \in L^1((0, \infty))$.

(2) Let (X, μ) be a finite measure space and let f be a nonnegative function in $L^1(X)$.
  (a) Show that $\sqrt{f} \in L^1(X)$.
  (b) Prove that if f_n is a sequence of nonnegative functions in $L^1(X)$ and $f_n \to f$ in L^1, then $\sqrt{f_n} \to \sqrt{f}$ in L^1.

(3) The Lebesgue outer measure of any subset E of $\mathbb{R}$ is

$$\mu^*(E) = \inf\{\sum_{k=1}^{\infty} \text{length}(I_k) \, : \, E \subset \cup_{k=1}^{\infty} I_k, \text{ where } I_k \text{ is an interval}\}.$$

(By interval we mean an open, closed or half-open interval of positive length.)
  (a) Prove that $\mu^*(\mathbb{Q} \cap [0, 1]) = 0$. ($\mathbb{Q}$ is the rational numbers.)
  (b) Let $\{A_n \, : \, n \in \mathbb{N}\}$ be a family of subsets of $\mathbb{R}$. Prove $\mu^*(\cup_n A_n) \leq \sum_n^{\infty} \mu^*(A_n)$.

(4) Let $1 < p \leq q < \infty$. Prove that $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^r(\mathbb{R})$ if and only if $p \leq r \leq q$.

(5) Let $\mathcal{H}$ be a separable complex Hilbert space and $V \subset \mathcal{H}$ a closed subspace. Assume that $y \in \mathcal{H}$, but $y \notin V$. Show that the subspace W spanned by V and y is closed in $\mathcal{H}$.

(6) Show that if f is a real valued function in $L^1([0, 1])$ and

$$\int_0^1 f(x) \exp(nx) \, dx = 0, \text{ for all } n \in \mathbb{Z},$$

then $f = 0$, a.e. (Hint: use Stone-Weierstrauss.)