Real Analysis Comprehensive Exam
Date: August 2014

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Regularity for Lebesgue measure m states that for any measurable set E

$$\sup_{K \subset E} m(K) = m(E) = \inf_{E \subset U} m(U)$$

where the K are compact and the U are open.
(a) Verify directly that this holds for the set $E = \mathbb{Q} \cap [0,1]$.
(b) Do the same for the set $E = (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$.
($\mathbb{Q}$ denotes the rational numbers.)

(2) Compute $\lim_{n\to\infty} \int_0^1 \cos^n(x^n)\, dx$ and carefully justify the computation. (Hint: You may want to show $1 - \cos^n(t) \le nt$, for $0 \le t$.)

(3) Prove that

$$\lim_{t\to 0+} \frac{1}{t^{1/2}} \int_0^t f(x)\, dx = 0,$$

for every $f \in L^2(\mathbb{R})$.

(4) Suppose that $f_n \in L^1(X, \mathcal{M}, \mu)$ for $n = 1, 2, \ldots,$ and $\sum_n \|f_n\|_1 < \infty$, show that $\sum_n f_n$ converges a.e.

(5) For $f \in L^\infty([0,1])$ define $Tf = \int_0^1 \int_0^t f(s)\, ds\, dt$.
(a) Show T is a bounded linear functional on $L^\infty([0,1])$.
(b) If $0 < a \le 1$, find $T(1_{[0,a]})$. Here $1_{[0,a]}$ is the indicator function, i.e., the characteristic function $\chi_{[0,a]}$.
(c) Because $C([0,1])$ is isometrically included in $L^\infty([0,1])$, T is a bounded linear functional on $C([0,1])$. Determine the measure μ on $[0,1]$ such that $Tf = \int f\, d\mu$ for all $f \in C([0,1])$. It is sufficient to compute $\mu((a,b])$ for all a, b with $0 \le a < b \le 1$.

(6) Consider the space of continuous functions $C([0,1])$ on the unit interval with the uniform norm. Set $B = \{f \in C([0,1]) \,:\, \|f\|_u \le 1\}$.
(a) True or false: There is a sequence of functions $f_n \in B$ which has no uniformly convergent subsequence.
(b) True or false: B is compact in the norm topology.
Give proofs for your answers (and be sure to give the statements of any key theorems you use).