Real Analysis Comprehensive Exam
Date: June 2014

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Suppose that A is an uncountable index set and $\{I_\alpha : \alpha \in A\}$ is a family of intervals in $\mathbb{R}$, that is, I_α is an interval (open, half-open, or closed) with positive length for each $\alpha \in A$. Prove that $\bigcup_{\alpha \in A} I_\alpha$ is Lebesgue measurable.

(2) For $a \in (0,1)$ and $f \in L^1([0,1])$ set

$$F(x) := \int_0^x \frac{1}{(x-y)^a} f(y)\, dy.$$

Show that $F(x)$ exists a.e. on $[0,1]$ and $F \in L^1([0,1])$.

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space.
(a) Define the completion of $(X, \mathcal{M}, \mu)$.
(b) Suppose $\mu = \frac{1}{2}(\delta_0 + \delta_1)$. Determine the completion of $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra of $\mathbb{R}$. Here δ_t is the Dirac (point) measure at t.
(c) Prove or disprove: There is a finite positive measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that the measure of every non-empty open interval is positive and the σ-algebra $\overline{\mathcal{M}}$ for the completion of the measure is the power set of $\mathbb{R}$.

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f \geq 0$ is an L^1 function so that $\int_X f^n\, d\mu = \int_X f\, d\mu$, for all $n \in \mathbb{N}$. Prove that f equals a characteristic (i.e., indicator) function a.e.

(5) Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove the following.
(a) If $f \in L^1(X)$ and $\epsilon > 0$, then

$$\int_X |f| d\mu \geq \epsilon\, \mu(\{x \in X : |f(x)| \geq \epsilon\}).$$

(b) For $1 \leq p < \infty$, if $f \in L^p(X)$, then

$$\lim_{t \to 0+} t^p \mu(\{x \in X : |f(x)| \geq t\}) = 0.$$

(6) (a) Prove that if (f_n) is a norm convergent sequence in $L^1([0,1])$ with limit f then (f_n) converges in measure to f.
(b) Prove or disprove: If (f_n) is a norm convergent sequence in $L^1([0,1])$ with limit f then (f_n) converges to f a.e.