Real Analysis Comprehensive Exam
Date: August 2013

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) For any closed interval $I = [a,b]$, define $|I| = b-a$. Consider outer measure on $\mathbb{R}$ defined by
$$\mu^*(S) = \inf\left\{\sum_{n=1}^{\infty} |I_n| \,:\, S \subset \cup_{n=1}^{\infty} I_n\right\},$$
where the I_n are closed intervals. Suppose $E, F \subset \mathbb{R}$ (not necesarily measurable).
 (a) Show that if $\operatorname{dist}(E,F) := \inf\{|x-y| \,:\, x \in E, y \in F\} > 0$, then $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$.
 (b) Suppose that E, F are disjoint, both are closed and one is compact, show that $\operatorname{dist}(E,F) > 0$.
(You may assume that μ^* is in fact an outer measure.)

(2) Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that f is integrable and let $E = \{x : f(x) \neq 0\}$. Prove that E is σ-finite.

(3) Fix $f \in L^1([0,1])$ and define a linear functional $T : L^\infty([0,1]) \to \mathbb{C}$ by
$$T(g) = \int_0^1 f(x)g(x)\,dx.$$
 (a) Calculate the norm of T.
 (b) Find a function $g \in L^\infty([0,1])$ so that $||g||_\infty = 1$ and $|T(g)| = ||T||$.

(4) Compute
$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\sin(x^n)}{x^n}\,dx.$$
Give full justification for your answer.

(5) Let m^2 denote Lebsgue measure on $\mathbb{R}^2$. Suppose $E \subset \mathbb{R}^2$ is a Borel set so that $E_t := \{(x,y) \in E \,:\, x = t\}$ is finite for all $t \in \mathbb{R}$. Prove that $m^2(E) = 0$.

(6) Let $S = \{x \in \mathcal{H} : ||x|| = 1\}$ be the unit sphere in a Hilbert space $\mathcal{H}$. Prove that if $\dim(\mathcal{H}) = \infty$, then S is noncompact.