Real Analysis Comprehensive Exam
Date: May 2012

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Let $f(x) = \sin(\frac{1}{x}), 0 < x < \infty$.
   (a) Show that $f \in L^2((0,\infty))$.
   (b) Prove
$$\lim_{n\to\infty} \int_0^\infty \left| \sin\left(\frac{n}{1+nx}\right) - \sin\left(\frac{1}{x}\right) \right|^2 dx = 0$$
   Be sure to justify all steps in your proof.

(2) Determine $\alpha_0 \in \mathbb{R}$ so that if $\alpha < \alpha_0$ then
$$\int_0^1 \frac{|f(x)|}{x^\alpha}\, dx < \infty,$$
holds for all $f \in L^3([0,1])$, while if $\alpha > \alpha_0$, then there is an $f \in L^3([0,1])$ so that
$$\int_0^1 \frac{|f(x)|}{x^\alpha}\, dx = \infty.$$

(3) Suppose that A is a measurable subset of $[0,1]$. Prove that there is a measurable $B \subset A$ so that $m(B) = \dfrac{1}{2} m(A)$. (Here, m is Lebesgue measure.)

(4) Let X be a Banach space. Recall that the open balls are the sets of the following form: $B_r(x_0) = \{x \in X\,;\, ||x - x_0|| < r\}$. Do the following:
   (a) Prove that the closure of any open ball $B_r(x_0)$ is
$$\overline{B_r(x_0)} = \{x \in X\,;\, ||x - x_0|| \le r\}.$$
   (We refer to $\overline{B_r(x_0)}$ as a closed ball.)
   (b) Prove that $\overline{B_r(x_0)} \subset \overline{B_s(y_0)}$ if and only if $||x_0 - y_0|| \le s - r$.
   (c) Prove that any decreasing sequence of closed balls has nonempty intersection.

(5) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose that $g \in L^1(X)$ and there exists $M \in \mathbb{R}$ so that
$$\left| \int_X \phi g \, d\mu \right| < M ||\phi||_1,$$
for all simple functions ϕ. Prove that $g \in L^\infty(X)$.

(6) Let λ and μ be positive measures on $(X, \mathcal{M})$. Define what it means for λ and μ to be mutually singular (written $\lambda \perp \mu$). Define what it means for λ to be absolutely continuous with respect to μ (written $\lambda \ll \mu$). Prove that if $\lambda \perp \mu$ and $\lambda \ll \mu$, then $\lambda = 0$.