Real Analysis Comprehensive Exam
Date: August 2011

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Calculate

$$\lim_{n\to\infty}\int_0^\infty \frac{x^{1/2}}{1+x^{2n}}\,dx.$$

Be sure to justify your answer.

(2) Prove that the sum of two sequences that converge in measure also converges in measure.

(3) Suppose $f:\mathbb{R}\to\mathbb{C}$ satisfies $|f(x)-f(y)|\le|x-y|$ for all $x,y\in\mathbb{R}$. Show that f is differentiable almost everywhere.

(4) Let $f:\mathbb{R}\to\mathbb{R}$ be a compactly supported continuous function. Prove that

$$\lim_{y\to 0}\int_{\mathbb{R}}|f(x+y)-f(x)|\,dx=0.$$

(5) Let $C([0,1]^2)$ denote the space of complex-valued continuous functions on $[0,1]^2 = [0,1]\times[0,1]$, with the norm $||f|| = \sup_{(x,y)\in[0,1]^2}|f(x,y)|$. Suppose that Λ is a bounded linear functional on $C([0,1]^2)$ and that $\Lambda f = 0$ for all $f\in C([0,1]^2)$ of the form $f(x,y)=\phi(x)\psi(y)$. Show that $\Lambda = 0$.

(6) Suppose that $(a_n)_{n=1}^\infty$ is a sequence of complex numbers such that $\sum_{n=1}^\infty a_n b_n$ converges for every sequence (b_n) satisfying $\sum|b_n|<\infty$. Show that the sequence (a_n) is bounded.