Real Analysis Comprehensive Exam
May 2011

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Suppose that $f \in L^1((0,\infty))$, and define $F(x)$ for $x > 0$ by

$$F(x) = \int_0^\infty f(t)e^{-xt}\,dt.$$

Show that F is differentiable on $(0,\infty)$, with

$$F'(x) = -\int_0^\infty tf(t)e^{-xt}\,dt.$$

(2) Suppose $f \in L^1(\mathbb{R})$ and f is uniformly continuous. Prove that $\lim_{x\to\infty} f(x) = 0$.

(3) Consider the region $\mathcal{R} = \{(x,y) : 0 < x < 1 \text{ and } 0 \le y < \infty\}$ in $\mathbb{R}^2$. Determine all real numbers b so that the function defined by

$$f(x,y) = \frac{e^{-y/x}}{x^b}$$

is an L^1 function on $\mathcal{R}$. Give full justification of your answer. (Note: you are to prove that the function is L^1 for certain values of b and not L^1 for the others.)

(4) Suppose that $(X,\mathcal{A},\mu)$ is a finite measure space, $f \in L^1(\mu)$, and $\mathcal{B}$ is a σ-algebra on X with $\mathcal{B} \subset \mathcal{A}$. Let ν denote the restriction of the measure μ to $\mathcal{B}$. Show that there exists an integrable $\mathcal{B}$-measurable function g such that

$$\int_E g\,d\nu = \int_E f\,d\mu$$

for all $E \in \mathcal{B}$.

(5) Suppose that μ is a complex Borel measure on $\mathbb{R}$ and $F : \mathbb{R} \to \mathbb{C}$ is defined by $F(x) = \mu((-\infty,x))$. Suppose that $\phi : \mathbb{R} \to \mathbb{C}$ is continuously differentiable with compact support. Show using Fubini's Theorem that

$$\int F(x)\phi'(x)\,dx = -\int \phi(x)\,d\mu(x).$$

(6) Find the smallest number p_0 so that the following statement holds for $p_0 < p$: $x^{-\frac{1}{5}}f(x) \in L^1([0,1])$, for all $f \in L^p([0,1])$. Prove your answer is correct.