Real Analysis Comprehensive Exam
January 2011

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Completely correct solutions to four of the six problems will guarantee a pass. Partial solutions may also be considered on their merit.

(1) Prove the following.

(a) $\lim_{n\to\infty} \int_0^1 \sin(\frac{x}{n}) \frac{n^3}{1+n^2 x}\, dx = 1$ and

(b) $\int_0^1 \sin(\frac{x}{n}) \frac{n^3}{1+n^2x^2}\, dx$ diverges as $n \to \infty$.

(2) Let μ be a measure on the measurable space $(X, \mathcal{A})$.

(a) Define what it means for a sequence of measurable functions f_n to converge in measure.

(b) Prove that if f_n converges in measure to both f and g, then $f = g$ a.e.

(3) Recall that part of the Lebesgue-Radon-Nikodym Theorem states that if λ is a σ-finite measure and μ is a positive measure, then λ has a Lebesgue decomposition with respect to μ. Show that this statement fails when the hypothesis that λ is σ-finite is omitted. Hint: consider the counting measure λ on $\mathbb{R}$.

(4) Suppose $(X, \mathcal{M}, \nu)$ is a measure space.

(a) Give the definition of $L^\infty(X)$ and the norm $||\cdot||_\infty$.

(b) Prove that when $1 < p < \infty$, $L^1(X) \cap L^\infty(X) \subset L^p(X)$.

(c) Prove that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$, for $f \in L^1(X) \cap L^\infty(X)$.

(5) Carefully prove that $\frac{\sin(x)}{x^\alpha} \in L^1(0,\infty)$ if and only if $1 < \alpha < 2$.

(6) Consider the space of continuous functions $C([0,1])$ on the unit interval with the uniform norm. Set $B = \{f \in C([0,1]) | \, \|f\|_u \leq 1\}$.

(a) True or false: There is a sequence of functions $f_n \in B$ which has no uniformly convergent subsequence.

(b) True or false: B is compact in the norm topology.

Give proofs for your answers (and be sure to give the statements of any key theorems you use).