Real Analysis Comprehensive Exam
August 2010

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Completely correct solutions to four of the six problems will guarantee a pass. Partial solutions may also be considered on their merit.

(1) Suppose f is a bounded measurable function on $\mathbb{R}$. Define
$$F(t) = \int_0^\infty f(x)e^{-tx}\,dx.$$
Show that F is continuous on $(0,\infty)$. Is F differentiable on $(0,\infty)$? Give full justifications. (dx refers to Lebesgue measure on $\mathbb{R}$.)

(2) Suppose f is a real valued L^1 function on $\mathbb{R}$ (with respect to Lebesgue measure μ) and $|\int_E f\,d\mu| \le \mu(E)$, for all measurable sets E. Prove that $|f(x)| \le 1$ for almost all x.

(3) Suppose $E \subset \mathbb{R}$ has Lebesgue measure zero. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is increasing and absolutely continuous, then $f(E)$ has Lebesgue measure zero.

(4) Let $1 < p \le q < \infty$. Determine exactly the values of r (satisfying $1 < r < \infty$) for which $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^r(\mathbb{R})$. Prove your assertion (this means prove the inclusion holds for certain values of r and does not hold for the other values).

(5) Suppose X and Y are two normed vector spaces and $T : X \to Y$ is a linear map. Recall that T is said to be *bounded* if and only if there is a constant C so that $||T(x)|| \le C||x||$ for each $x \in X$. Prove that the following are equivalent.
  (a) T is continuous (at each $x \in X$).
  (b) T is continuous at 0.
  (c) $\sup_{||\eta||=1} ||T(\eta)||$ is finite.
  (d) T is bounded.
  (e) T is uniformly continuous.

(6) Suppose F is a continuous function on $[0,1] \times [0,1]$. Prove that for any $\epsilon > 0$, there exist N, $a_i, b_i \in C([0,1])$, for $i = 1, 2, \ldots, N$ so that
$$|F(x,y) - \sum_{i=1}^{N} a_i(x)b_i(y)| < \epsilon,$$
for all $x, y \in [0,1]$.