Real Analysis Comprehensive Exam
January 2010

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. Complete solutions to four of the six problems will guarantee a pass.

(1) Prove or disprove each of the following statements.
(a) $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$.
(b) $L^1([0,1]) \subset L^2([0,1])$.
(c) $L^2([0,1]) \subset L^1([0,1])$.
(d) $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$.

(2) Let $(X, \mathcal{A}, \mu)$ be a measure space and let f_n be a sequence of measurable functions on X. Suppose that

$$\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x)| \geq 1/m\})$$

converges for all natural numbers m. Prove that $f_n(x) \to 0$ almost everywhere.

(3) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f_n be a sequence of positive measurable functions on X. Suppose that $f_n \to f$ pointwise and that, for each n, $f_n(x) \leq f(x)$ almost everywhere. Prove that

$$\lim_{n \to \infty} \int_X f_n = \int_X f.$$

(Hint: Consider $g_n(x) = \inf_{k \geq n} \{f_k(x)\}$.)

(4) Consider a σ-finite measure space (X, μ). Fix a function $K \in L^2(X \times X)$ and define a linear map

$$T(f) : L^2(X) \to L^2(X) \text{ by } T(f)(x) = \int_X K(x,y) f(y) \, d\mu(y). \tag{A}$$

Give a careful proof that T is well-defined (i.e., the integral in (A) converges for almost all x and defines a function in $L^2(X)$) and T is bounded with operator norm satisfying $||T|| \leq ||K||_2$.

(5) Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{y \to 0} \int_{\mathbb{R}} |f(x+y) - f(x)| dx = 0.$$

(Hint: First assume f is a compactly supported continuous function.)

(6) Prove that $C([0,1], \mathbb{R})$, the space of real-valued continuous functions on the unit interval (with the uniform norm), is separable (i.e., contains a countable dense subset).