Real Analysis Comprehensive Exam
August 2009

Give complete and grammatically correct solutions. If you use a standard theorem, then you must state that theorem and explicitly verify the hypothesis. For example, if you use a form of the Dominated Convergence Theorem, then you must state the relevant version and verify its hypothesis. Completely correct solutions to four of the six problems will guarantee a pass. Partial solutions may also be considered on their merit.

(1) Define a function F by the formula

$$F(y) = \int_0^\infty e^{iyx^2} \frac{1}{(1+x^2)^\alpha}\, dx$$

when $\alpha \geq 1$. Prove:

(a) F is a continuous function on $\mathbb{R}$ and is bounded by $\frac{\pi}{2}$.

(b) $F'(y)$ exists when $\alpha > 3/2$.

(2) Suppose that $\mathcal{A}_i$ is a σ-algebra on X_i for $i = 1, 2$. Denote the projection of $X = X_1 \times X_2$ onto X_i by $\pi_i : X \to X_i$, $i = 1, 2$. Define the product σ-algebra $\mathcal{A}$ on X to be the σ-algebra generated by

$$\{\pi_i^{-1}(E_i) : E_i \in \mathcal{A}_i, i = 1, 2\}.$$

Prove the following.

(a) If $\mathcal{S}_i$ generates $\mathcal{A}_i$, for $i = 1, 2$, then the product σ-algebra is generated by

$$\{\pi_i^{-1}(E_i) : E_i \in \mathcal{S}_i, i = 1, 2\}. \qquad \text{(A)}$$

(Hint: Let $\mathcal{A}'$ be the σ-algebra generated by (A) and, for each $i = 1, 2$, consider $\{E_i : \pi_i^{-1}(E_i) \in \mathcal{A}'\}$.)

(b) Suppose $\mathcal{B}$ is a σ-algebra on a set Y and $f : Y \to X$. Then f is measurable if and only if $\pi_i \circ f$ is a measurable function from Y to X_i, for $i = 1, 2$.

(3) Consider the region $\mathcal{R} = \{(x, y) : 0 < x < 1 \text{ and } 0 \leq y < \infty\}$ in $\mathbb{R}^2$. Is the function defined by $f(x, y) = \dfrac{e^{-y/x}}{x^{4/3}}$ in $L^1(\mathcal{R})$? Give full justification of your answer.

(4) Consider a signed measure λ and a (positive) measure μ on the measurable space $(X, \mathcal{A})$.

(a) Define what it means for λ and μ to be mutually singular.

(b) Define what it means for λ to be absolutely continuous with respect to μ.

(c) Suppose that λ and μ are mutually singular and λ is absolutely continuous with respect to μ. Prove that $\lambda = 0$.

(d) Give an example of a measure $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure. (Your example should not be the zero measure or a constant multiple of Lebesgue measure.) You need not prove your assertion.

(5) Let $\mathcal{H}$ be a Hilbert space.

(a) Give the definition of weak convergence of a sequence (x_n) in $\mathcal{H}$.

(b) Give an example of a sequence in some Hilbert space $\mathcal{H}$ that converges weakly, but does not converge in the norm topology. Prove your assertions.

(c) Prove that if a sequence (x_n) converges weakly to x and $||x_n|| \to ||x||$, then $x_n \to x$ in the norm topology.

(6) Suppose f is a continuous function on $[0, 1]$ and $\int_0^1 f(x)\, x^n\, dx = 0$, for all $n = 0, 1, 2, \ldots$. Prove that $f = 0$.