Provide complete proofs to all your assertions except where explicitly instructed otherwise. All notation and terminology should be clearly defined and all proofs should be complete and grammatically correct. If you require a standard result, then state it before you use it. Five problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Prove that if $E \subset [0,1]$ has Lebesgue measure equal to zero, then the complement of E is dense in $[0,1]$.

2. Prove or disprove: The product of two absolutely continuous functions on $[a,b]$ is absolutely continuous.

3. (a) Carefully state the Monotone Convergence Theorem, Fatou's Lemma, and the Lebesgue Dominated Convergence Theorem.

   (b) Give an example of a uniformly bounded sequence $\{f_k(x)\}$ of Lebesgue-integrable functions $f_k : \mathbb{R} \to \mathbb{R}$ that converges pointwise to a Lebesgue-integrable function $f(x)$ for all $x \in \mathbb{R}$ and satisfies:
   $$\lim_{k\to\infty} \int_{\mathbb{R}} f_k(x) \text{ exists and does not equal } \int_{\mathbb{R}} f(x).$$

   (c) Give an example of a sequence $\{f_k(x)\}$ of Lebesgue-integrable functions $f_k : [0,1] \to \mathbb{R}$ that converges pointwise to a Lebesgue-integrable function $f(x)$ for all $x \in [0,1]$ and satisfies:
   $$\lim_{k\to\infty} \int_0^1 f_k(x) \text{ exists and does not equal } \int_0^1 f(x).$$

4. (a) Carefully state a version of Fubini's Theorem that applies to integrals of the form $\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy$ where $dx\,dy$ is Lebesgue-measure on $\mathbb{R}^2$.

   (b) Show that the conclusion of Fubini's theorem does not hold for the integral:
   $$\int_0^1 \int_0^\infty (2-xy)xye^{-xy}\,dy\,dx.$$

5. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space.

   (a) Give a careful definition of $L^\infty(X,\mu)$. Be sure to include the definition of the L^∞ norm.

   (b) Let $1 \le p < \infty$. Prove that if $\mu(X) < \infty$ then there exists a constant $C > 0$ so that
   $$||f||_p \le C\,||f||_\infty$$
   for all $f \in L^\infty(X,\mu)$.

6. Show that $L^1([0,1])$ is separable.