Real Analysis Comprehensive Exam
August 2008

Do all six problems. Explicitly verify the hypotheses of all major theorems used. In problems 2, 4 and 5, dx refers to Lebesgue measure.

(1) Let $\{E_n : n = 1, 2....\}$ be a family of measurable sets in a measure space $(X, \mathcal{M})$. Show that the set $A = \{x : x \text{ is in all but finitely many } E_n\}$ is measurable.

(2) Prove or disprove: $\dfrac{\sin(x)}{x} \in L^1((0, \infty), dx)$.

(3) Give the definition of an absolutely continuous function on $I = [0, 1]$. Prove that if f is absolutely continuous on I and $f > 0$, then $1/f$ is absolutely continuous on I.

(4) Calculate

$$\lim_{n \to \infty} \int_0^\infty \frac{x^{1/2}}{1 + x^{2n}}\, dx.$$

Be sure to justify your answer.

(5) Assume that $f_n, f \in L^1([0, 1])$.

(a) If f_n converges to f in L^1-norm as $n \to \infty$, then prove that there exists a subsequence f_{n_j} so that f_{n_j} converges to f almost everywhere.

(b) Assume now that f_n converges to f almost everywhere. Prove that f_n converges to f in measure.

(6) Let $p = 3$ and (X, μ) be any measure space. Prove that

$$\int_X |fgh|\, d\mu \leq ||f||_p ||g||_p ||h||_p,$$

for $f, g, h \in L^p(X, \mu)$.