Real Analysis Comprehensive Exam
August 2007

Explicitly verify the hypotheses of all major theorems used. The measure m in Problem 1 is Lebesgue measure on $[0,1]$. The measure dy in Problem 4 is Lebesgue measure.

(1) Suppose $E_1 \subset E_2 \subset \ldots$ is an increasing sequence of measurable subsets of $[0,1]$. Set $E = \cup_n E_n$. Prove that for every $\epsilon > 0$ there exists an N such that $m(E) < m(E_n) + \epsilon$ for every $n \geq N$.

(2) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f_n be a sequence of positive measurable functions on X. Suppose that $f_n \to f$ pointwise and that, for each n, $f_n(x) \leq f(x)$ almost everywhere. Prove that

$$\lim_{n\to\infty} \int_X f_n = \int_X f.$$

(Hint: Consider $g_n(x) = \inf_{k \geq n} \{f_k(x)\}$.)

(3) Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \exp(-\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Is f absolutely continuous on $[0,1]$?
(b) Is f of bounded variation on $[0,1]$?
Prove your answers. (Hint: Is f differentiable?)

(4) Let $f_\alpha(x,y) = (1 + x^2y^2)^\alpha$, for $\alpha, x, y \in \mathbb{R}$.
(a) For each $\alpha \in \mathbb{R}$ determine the set U_α of all real numbers x such that $\int_1^\infty |f_\alpha(x,y)|\, dy < \infty$.
Now, define $F_\alpha : U_\alpha \to \mathbb{R}$ by

$$F_\alpha(x) = \int_1^\infty f_\alpha(x,y)\, dy. \tag{0.1}$$

(b) For each α determine all values of $x \in U_\alpha$ for which F_α is continuous at x.
(c) For each α determine all values of $x \in U_\alpha$ for which F_α is differentiable at x.
Give complete proofs of your answers; be sure to verify the hypotheses of any theorems you use.

(5) Suppose F is a continuous function on $[0,1] \times [0,1]$. Prove that for any $\epsilon > 0$, there exist N, $a_i, b_i \in C([0,1])$, for $i = 1, 2, \ldots, N$ so that

$$|F(x,y) - \sum_{i=1}^N a_i(x) b_i(y)| < \epsilon,$$

for all $x, y \in [0,1]$.

(6) Let $b = (b_n) \in \ell^2$. For $a = (a_n) \in \ell^2$, define $T_b(a)$ to be the sequence $(b_n a_n)$. Prove that T_b is a well-defined linear map $T_b : \ell^2 \to \ell^1$. Show that T_b is bounded and compute the operator norm of T_b.