Real Analysis Comprehensive Exam

January 2007

Explicitly verify the hypotheses of all major theorems used. The measure on $\mathbb{R}$ (and on $[0,1]$) is assumed to be Lebesgue measure.

(1) Consider functions $\mathbb{R} \to \mathbb{R}$. Determine whether each of the following two statements is true or is false. Give proof or counterexample.

(a) There exist functions f and g, so that f is not continuous at any point, g is continuous everywhere, but $f = g$ a.e.

(b) There exist continuous functions f and g, so that $f = g$ a.e., but $f \neq g$.

(2) Let λ and μ be positive measures on $(X, \mathcal{M})$. Define what it means for λ and μ to be mutually singular (written $\lambda \perp \mu$). Define what it means for λ to be absolutely continuous with respect to μ (written $\lambda \ll \mu$). Prove that if $\lambda \perp \mu$ and $\lambda \ll \mu$, then $\lambda = 0$.

(3) Let (X, μ) be a measure space. Give the definition of $L^\infty(X, \mu)$. Include a definition of $|| \; ||_\infty$. Prove that if f_n is a sequence in $L^\infty(X, \mu)$ and $||f_n||_\infty \to 0$, then there exists a set E of measure zero so that $f_n \to 0$ uniformly on the complement of E.

(4) Define a function by

$$F(x) = \int_0^\infty e^{-xy} \frac{1}{1+y^2} \, dy.$$

Prove that $F(x)$ is well defined for $x \geq 0$ and is differentiable for $x > 0$.

(5) Consider the space of continuous functions $C([0,1])$ on the unit interval with the uniform norm. Set $B = \{f \in C([0,1]) | \, \|f\|_u \leq 1\}$.

(a) True or false: There is a sequence of functions $f_n \in B$ which has no uniformly convergent subsequence.

(b) True or false: B is compact in the norm topology.

Give a proof for your answers AND give the statements of any key theorems you use.

(6) Let S be a subspace of $L^2([0,1])$ that is closed with respect to the L^1 norm.

(a) Show that S is closed with respect to the L^2 norm.

(b) Show that there exists a constant $M > 0$ so that $\|f\|_2 \leq M\|f\|_1$, for all $f \in S$. (Hint: apply the open mapping theorem.)