Real Analysis Comprehensive Exam

August 2006

Do all five problems. Explicitly verify the hypotheses of all major theorems used.

(1) Consider Lebesgue measure μ on $X = [0,1]$. State whether the following are TRUE or FALSE for measurable sets E. Give a proof if the statement is TRUE and a counterexample if FALSE.

(a) $\mu(E) > 0$ implies E is uncountable.

(b) $\mu(E) = 0$ implies E is countable.

(c) $\mu(E) = 1$ implies E is dense in X.

(d) $\mu(E) > 0$ implies the the interior of E is nonempty.

$[0,2\pi]$?? (2) Suppose that $f : [a,b] \to \mathbb{C}$.

(a) Suppose f is continuous and

$$\int_a^b e^{inx} f(x)\, dx = 0, \text{ for all integers } n.$$

Prove that f is identically zero.

(b) What can you say if f is an L^1 function satisfying

$$\int_a^b e^{inx} f(x)\, dx = 0, \text{ for all integers } n?$$

Give a proof that your answer is correct.

(3) Consider $\mathbb{R}$ with Lebesgue measure. Suppose $f \in L^1(\mathbb{R})$ and

$$F(x) = \int_{-\infty}^{\infty} f(t) \sin(x/t)\, dt, \text{ for } x \in \mathbb{R}.$$

(a) Prove that $F(x)$ is a continuous function on $\mathbb{R}$.

(b) Prove that when $f(t) = \dfrac{t}{(1+t^2)^\alpha}$, then $F(x)$ is differentiable for all $x \in \mathbb{R}$ when $\alpha > 1$.

(4) Let $1 \le p \le \infty$. Recall that $\ell^p = L^p(\mathbb{N}, \lambda)$, where λ is the counting measure. Therefore, when $1 \le p < \infty$, ℓ^p consist of sequences $a = (a_n)$ satisfying $\sum |a_n|^p < \infty$ and ℓ^∞ consists of bounded sequences. For two sequences a and b define $T_b(a)$ to be the sequence $(a_n b_n)$.

(a) Describe exactly those p_1 and p_2 for which $\ell^{p_1} \subset \ell^{p_2}$ holds. Prove your answer.

(b) Let $b = (b_n)$ be in ℓ^∞. Prove that T_b gives a well-defined bounded operator from ℓ^p to ℓ^p, for $1 \le p \le \infty$. Compute the norm of T_b.

(c) Fix p with $1 \le p < \infty$. Describe exactly those r with $1 \le r < \infty$ so that T_b gives a well-defined bounded linear map $\ell^r \to \ell^1$ for *every* $b \in \ell^p$. Prove your answer.

(5) Suppose X is a locally compact Hausdorff space. Define what it means for ν to be a Radon Measure on X. Prove that if ν is a finite Radon measure on X, then for any Borel set E in X and every $\epsilon > 0$, there is a continuous function f on X so that $f(x) \ge 1$ for all $x \in E$ and

Compact??

$$|\nu(E) - \int_X f\, d\nu| < \epsilon.$$