Real Analysis Comprehensive Exam

August 2004

Do all six problems. Explicitly verify the hypotheses of all major theorems used.

(1) Suppose that f is Borel measurable on $\mathbb{R}$. Prove that $g(x,y) = f(x-y)$ is Borel measurable on $\mathbb{R}^2$.

(2) Suppose $(X, \mathcal{M}, \mu)$ is a measure space.

(a) Give a careful definition of $L^\infty(X,\mu)$. Be sure to include the definition of the L^∞ norm.

(b) Let $1 \leq p < \infty$. Prove that if $\mu(X) < \infty$ then there exists a constant $C > 0$ so that

$$||f||_p \leq C||f||_\infty$$

for all $f \in L^\infty(X,\mu)$.

(3) Suppose that $f : X \to R$ is measurable with respect to the measure space $(X, \mathcal{M}, \mu)$ and, for all $x \in X$, $0 \leq f(x) \leq 1$. Let $E = f^{-1}(1)$. For all natural numbers n define the function $f^n : X \to R$ by $f^n(x) = (f(x))^n$.

(a) Prove that

$$\lim_{n\to\infty} \int f^n \, d\mu = \mu(E).$$

if μ is a finite measure.

(b) Prove that, for any non-negative measurable function $g : X \to R$,

$$\lim_{n\to\infty} \int (1 - f^n) g \, d\mu = \int_{X-E} g \, d\mu$$

for any measure μ, finite or infinite.

(4) Let

$$f(t) = \int_0^\infty \frac{\sin xt}{x(1+x^2)} \, dx,$$

for $t \in \mathbb{R}$. Prove that f is differentiable and find f'.

(5) Characterize the set of functions f such that f is continuous on $[-1,1]$ and $\int_{-1}^1 f(x)\, x^{2k}\, dx = 0$ for $k = 1, 2, \ldots$. Prove your answer.

(6) Suppose $(X, \mathcal{M}, \mu)$ is a measure space (with μ a non-negative measure) and $f : X \to \mathbf{R}$ a Borel measurable function. Define μ_f on the Borel sets by

$$\mu_f(E) = \mu(f^{-1}(E)),$$

for any Borel set $E \subset \mathbf{R}$.

(a) Show that μ_f is a measure.

(b) Prove that for any nonnegative Borel function $g : \mathbf{R} \to \mathbf{R}$,

$$\int g \, d\mu_f = \int g \circ f \, d\mu.$$

(Hint: Consider simple functions first.)