Real Analysis Comprehensive Exam

August 2003

Do all six problems. Explicitly verify the hypotheses of all major theorems used.

(1) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_j : X \to \mathbf{R}, j = 1, 2, ...$ be a family of measurable functions. Let $f(x) = \sup_j\{f_j(x)\}$ (you may assume that $f(x) \in \mathbf{R}$, for all $x$). Give a careful proof that $f$ is measurable.

(2) (a) For a function $f : [a, b] \to R$ define absolutely continuous.

(b) Show directly from the definition that if $f : R \to R$ is such that $f'$ is continuous, then $f$ is absolutely continuous on any finite interval $[a, b]$.

(3) Prove that

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx = 1.$$

(4) Suppose that $f$ is a bounded continuous function on $[0, \infty)$ and that $\int_0^\infty f(x)\, e^{-nx}\, dx = 0$ for all $n \in \mathbb{N}$. Prove that $f = 0$.

(5) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbf{R}$ a nonnegative measurable function. For measurable sets $E$ define

$$\mu_f(E) = \int_E f(x)\, d\mu(x).$$

It is a fact that $\mu_f$ is a measure.

(a) Show that for a nonnegative measurable function $g : \mathbf{R} \to \mathbf{R}$

$$\int g(x)\, d\mu_f(x) = \int g(x)f(x)\, d\mu(x).$$

(Hint: Consider simple functions first.)

(b) Suppose that $f$ is bounded. Show that $L^1(X, d\mu) \subset L^1(X, d\mu_f)$.

(c) Show that the preceding statement may fail if $f$ is not bounded.

(6) Let $\mu$ be Lebesgue measure on $\mathbf{R}$. For a measurable set $E$ we say that the density of $E$ is well defined if

$$\lim_{T\to\infty} \frac{\mu(E \cap [-T, T])}{2T}$$

exists. If the density is well defined we set $D(E)$ equal to this limit and call it the *density* of $E$. Suppose that $E$ and $E_j$ below have well defined densities.

(a) Show that $D(E) \leq 1$ and $D(\text{bounded set}) = 0$.

(b) Prove or give a counterexample to the statement: if $r$ is a positive real number and $rE = \{rx : x \in E\}$ then $D(rE) = D(E)$.

(c) Show: $D(E_1 \cup E_2) = D(E_1) + D(E_2)$ when $E_1$ and $E_2$ are disjoint.

(d) Show that there are disjoint sets $\{E_j\}_{j=1,2,...}$ so that $E_j$ and $\cup_j E_j$ have well defined densities but $D(\cup_j E_j) \neq \sum_{j=1}^\infty D(E_j)$.