Comprehensive Exam
Real Analysis
January 2003

1. Consider the right-continuous function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x + [x]}{2}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Here $[x]$ is the greatest integer less than or equal to x. Let μ be the corresponding Lebesgue-Stieltjes measure on $\mathbb{R}$.

   (a) Find the measure of the following sets: $(0, 5), [0, 5]$ and $\{5\}$.

   (b) Give in explicit form the Lebesgue decomposition of μ with respect to Lebesgue measure on $\mathbb{R}$.

2. Consider the counting measure ν on $\mathbb{N}$ and let $\ell_1 = L_1(\mathbb{N}, \nu)$. Let $c_0 = \{a : \mathbb{N} \to \mathbb{C} \mid \lim_{n\to\infty} a(n) = 0\}$. Prove, from the definitions, that $b \mapsto \phi_b$ where $\phi_b(a) = \int_{\mathbb{N}} b(n)a(n)\, d\nu(n)$, is an isometry from ℓ_1 onto c_0^*.

3. Let $1 \leq p, q \leq \infty$ and $0 < a < 1$. Determine all of p, q and a so that $L_p([0, 1], \mu_a) \subset L_q([0, 1], \mu_a)$, where μ_a is the measure satisfying $\mu_a(B) = \displaystyle\int_B \frac{1}{x^a} dx$ for all Lebegue measurable sets B.

4. Prove the following two statements.

   (a) If $f \in L_1([0, 1])$ then for each $\epsilon > 0$ there exists a continuous function g on $[0, 1]$ so that $||f - g|| < \epsilon$.

   (b) Given $\epsilon > 0$, there exists an $f \in L_\infty([0, 1])$ so that $||f - g||_\infty \geq \epsilon$ for every $g \in C([0, 1])$.

5. Let $f_{m,n}(x) := (1/\sqrt{x} - x^2)(1 + \cos^m(\pi n! x))$, $m, n \in \mathbb{N}$. Show that the function

$$f(x) := \lim_{n\to\infty} \lim_{m\to\infty} f_{m,n}(x), \quad \forall\, x \in (0, 1),$$

   is Lebesgue integrable on $(0, 1)$ with respect to the measure dx. Find

$$\int_0^1 f(x)\, dx \quad \text{and} \quad \lim_{n\to\infty} \lim_{m\to\infty} \int_0^1 f_{m,n}(x)\, dx.$$

6. Let H be a real Hilbert space with an inner product $\langle x, y \rangle$ and the induced norm $||x|| := \sqrt{\langle x, x \rangle}$.

(a) Prove the Parallelogram Law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \forall\, x, y \in H.$$

(b) Suppose that B is a real Banach space where the Parallelogram Law holds. Define an inner product in B from the norm, and show that it satisfies the homogeneity condition. (Hint: First show that $\langle nx, y\rangle = n\langle x, y\rangle$ for any natural number n.)