Comprehensive Exam
Real Analysis
August 2002

1. Suppose $(X, \mathcal{M}, \mu)$ is a $\sigma-$finite measure space and let $1 < p < \infty$. It follows from the Hölder inequality that if $\mu(X) < \infty$ then $L_p(X, \mu) \subset L_1(X, \mu)$. Prove the converse: if $L_p(X, \mu) \subset L_1(X, \mu)$ then $\mu(X) < \infty$.

2. Let (e_n) be an orthonormal sequence in a separable Hilbert space H.

   (a) Show that there is an orthonormal basis for H which contains $\{e_n : n = 1, 2, \dots\}$.

   (b) Show that $e_n \to 0$ weakly, i.e., for any $h \in H$, $\langle h, e_n \rangle \to 0$.

3. Suppose (q_n) is an enumeration of the rational numbers in the interval $[0, 1]$. Define

$$F(x) = x + \sum_{q_n \leq x} \frac{1}{2^n}, x \in [0, 1].$$

   (a) Show that F is monotone.

   (b) Show that F is continuous at $x = 0$ and the irrationals and is discontinuous at each rational greater than 0.

   (c) What is the Lebesgue-Stieltjes measure generated by F, i.e., what is the measure ν such that

$$\int_0^1 g \, d\nu = \int_0^1 g \, dF$$

   for all $g \in C([0, 1])$. Justify your answer.

4. Show that $L_1([0, 1])$ is separable.

5. Compute

$$\int_0^\infty \int_0^1 \frac{e^{-\frac{y}{x}}}{x^{\frac{11}{13}}} \, dx dy.$$

   Give full justification for your computation.

6. Suppose that $f_n \to f$ in $L_2([0, 1])$, as $n \to \infty$. Determine whether the following statements are correct (give a proof if the answer is positive or a counterexample if negative):

   (a) $f_n \to f$ in $L_1([0, 1])$, as $n \to \infty$.

   (b) $f_n \to f$ a.e. on $[0, 1]$, as $n \to \infty$.

   (c) There is a subsequence n_k such that $f_{n_k} \to f$ a.e. on $[0, 1]$, as $n_k \to \infty$.