Comprehensive Exam
Real Analysis
August 2001

A passing mark will be given if 5 problems are completely solved.

1. Define $f_n \in L^1([0,2\pi])$ by $f_n(x) = e^{inx}$ $(n = 0, 1, 2, \ldots)$; similarly define $g \in L^1([0,2\pi])$ by $g(x) = e^{-ix}$.
   (a) Find a continuous linear functional $\lambda : L^1([0,2\pi]) \to \mathbb{C}$ such that $\lambda(g) \neq 0$ while $\lambda(f_n) = 0$ for $n = 0, 1, 2, \ldots$.
   (b) Find a number $\epsilon > 0$ such that

$$||g - \phi||_1 > \epsilon$$

   if ϕ is any linear combination of the functions f_n $(n = 0, 1, 2, \ldots)$. (State explicitly what ϵ is and prove that it works.)

2. Let X be a topological space.
   (a) Define the *Borel* σ-algebra of X.
   (b) Suppose $\{x_n : n = 1, 2, \ldots\}$ is a sequence of real numbers which is dense in $\mathbb{R}$. Prove or give a counterexample:
   If $f : X \to \mathbb{R}$ and $\{x : f(x) < x_n\}$ is Borel measurable for each n then f is Borel measurable.
   (c) Suppose $X = \mathbb{R}^n$. Is there a countable family of sets $(A_n)_{n=1}^{\infty}$ which generates the Borel σ-algebra of X?

3. (a) Give an example of $f : [0,1] \to \mathbb{R}$ such that f has bounded variation, f is almost everywhere differentiable and $|f'| \leq 1$ almost everywhere, but

$$f(1) - f(0) \neq \int_0^1 f'(t)\, dt.$$

   (b) Suppose that $f : [0,1] \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y|$$

   for all $x, y \in [0,1]$. Show that

$$f(1) - f(0) = \int_0^1 f'(t)\, dt.$$

4. Find
$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x^{2n})}{x^{2n}}\, dx.$$
Justify your answer.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) Show that if $\mu(X) < \infty$ then $L^2(\mu) \subset L^1(\mu)$.
   (b) Show that if $X = \mathbb{R}$ and μ =Lebesgue measure then $L^1(\mu) \not\subseteq L^2(\mu)$ and $L^2(\mu) \not\subseteq L^1(\mu)$.

6. Suppose that $(x_n)_{n=1}^\infty$ is a sequence of norm–one elements of ℓ_2 and that $(x_n(i))_{n=1}^\infty$ converges to $y(i)$ for each $i \in \mathbb{N}$.
   (a) Prove that y is in ℓ_2.
   (b) Show that if $\|y\|_2 = 1$ then $(x_n)_{n=1}^\infty$ converges to y in norm.