Comprehensive Exam
Real Analysis
August 2000

A passing mark will be given if 5 problems are completely solved.

1. Let $X = \mathbb{N} \times \mathbb{N}$ and ν be counting measure on X. Any real-valued function on X is measurable and defines a double sequence $(a_{n,m})_{n=1}^{\infty}{}_{m=1}^{\infty}$ by $a_{n,m} = f(n,m)$. Using the definition of the Lebesgue integral, state in terms of the double sequence exactly what it means for f to be integrable.

2. Let m be Lebesgue measure on $\mathbb{R}$, normalized so that $m([0,1]) = 1$. Let $F(x) = n$ if $n \leq x < n+1$ and n is an integer. (The greatest integer less than or equal x.) Let δ_t denote the dirac measure supported at t, i.e., $\delta_t(A)$ is 1 if $t \in A$ and 0 otherwise.
   (a) Define *Radon-Nikodym derivative* of a measure μ with respect to a measure ν and state a general sufficient condition for its existence.
   (b) Describe the Lebesgue-Stieltjes measure μ_F induced by $F(x)$.
   (c) Let $\nu = m + \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_n$. Find the Lebesgue decomposition of μ_F with respect to ν.

3. Recall than a sequence of measurable functions (f_n) on a measure space $(X, \mathcal{A}, \mu)$ is said to converge in measure to a function f if for every $\epsilon > 0$ there is a $\delta > 0$ and $N \in \mathbb{N}$ such that $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \delta$ for all $n \geq N$. Prove that if $(f_n)$is a sequence of integrable functions which converges to f in $L_1(X, \mathcal{A}, \mu)$, then (f_n) converges to f in measure. Must (f_n) also converge to f a.e. μ? Prove it or give a counterexample.

4. Let (γ_n) be a sequence of complex numbers and define an operator T by $T((a_n)_{n=1}^{\infty}) = (\gamma_n a_n)_{n=1}^{\infty}$, for any sequence of complex numbers $(a_n)_{n=1}^{\infty}$.
   (a) Show that T defines a bounded operator on ℓ_2 if and only if $(\gamma_n)_{n=1}^{\infty}$ is bounded.
   (b) If T defines a bounded operator on ℓ_2 give an explicit formula in terms of the sequence $(\gamma_n)_{n=1}^{\infty}$ for its norm.
   (c) Show that if $T((a_n)_{n=1}^{\infty}) \in \ell_2$ for all $(a_n)_{n=1}^{\infty} \in \ell_2$ then $(\gamma_n)_{n=1}^{\infty}$ is bounded.

5. Determine the closure of $\{\sum_{k=1}^{n} a_k[\sin(\frac{\pi(1-t)}{2})]^k : n \in \mathbb{N}, (a_k) \subset \mathbb{R}\}$ in $C[0,1]$ with the norm $\|f\| = \sup\{|f(t)| : t \in [0,1]\}$. Carefully justify your answer.

6. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ and for each $x, y \in \mathbb{R}$, let $f_x(y) = f(x,y)$ and $f^y(x) = f(x,y)$. If f_x is Borel measurable and f^y is continuous, prove that f is Borel measurable. (Hint: Find a sequence of functions of the form

$$\sum \left(\frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y) \right) 1_{A_i}$$

which converges to f, where $1_A(x) = 1$ if $x \in A$, 0 otherwise.)