PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & PDE—August 2006

Part I: Fourier Analysis

1. (a) Consider the function f defined on the interval $0 \leq x \leq \pi$ by $f(x) = 1$ for $0 \leq x \leq \pi/2$ and $f(x) = 0$ for $\pi/2 < x \leq \pi$. Expand f in a Fourier sine series.

   (b) At what points of the interval $-\pi \leq x \leq \pi$ does the sine series from part (a) converge, and to what does it converge at those points? Explain.

2. Be sure to give in each part a brief reason for your answer.

   (a) In each case determine for which x in the interval $-\pi \leq x \leq \pi$ the given trigonometric series converges.

   (i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} \sin nx$

   (ii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} \cos nx$

   (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^n n}{n^2} \sin nx$

   (b) Is the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} \sin nx$ the Fourier series of a square integrable function on the interval $-\pi \leq x \leq \pi$?

3. (a) If f is continuous with period 2π on the real line, does it follow that the Fourier series of f converges to f at each point? If so, briefly explain why. If not, give an additional condition ensuring convergence.

   (b) Let f be a continuous function with period 2π on the real line. Assume that the Fourier series of f converges uniformly to some function. Does it follow that the Fourier series of f converges uniformly to f? Give a brief justification for your answer.

   (c) Let f and g be square integrable functions on the interval $-\pi \leq x \leq \pi$ with the same sequence of Fourier coefficients. What can you conclude about f and g? Give a brief justification for your answer.

   (d) Let f and g be integrable functions on the interval $-\pi \leq x \leq \pi$ with the same sequence of Fourier coefficients. What can you conclude about f and g? (Assume that they are continuous except at finitely many points.) Give a brief justification for your answer.

Part II: Partial Differential Equations

1. Let $U \subset \mathbf{R}^n$ be open and bounded. We say $u \in C^2(\bar{U})$ is subharmonic if

$$-\Delta u \leq 0 \quad \text{in} \quad U.$$

Prove that if u is subharmonic then

$$u(x) \leq \fint_{\partial B(x,r)} u(y)dS(y) \quad \text{for all balls } B(x,r) \subset U.$$

2. Use the Fourier transform to find a formula for the solution to the convection-diffusion equation

$$\begin{cases} u_t - 4u_x - u_{xx} = 0, & x \in \mathbf{R},\, t > 0, \\ u(x,0) = f(x), & x \in \mathbf{R}. \end{cases}$$

Hint: The Fourier transform of a function u is defined by

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\xi\cdot x}\, u(x)\, dx,$$

and the inverse Fourier transform is defined by

$$u^{\vee}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\xi\cdot x}\, u(\xi)\, d\xi.$$

You may use the formula

$$(e^{-(\xi+a)^2 t})^{\vee} = \sqrt{\frac{1}{2t}} e^{-\frac{x^2}{4t}} e^{-iax} \quad \text{for any parameters } t > 0 \text{ and } a.$$

3. (a) Write d'Alembert's formula for the solution of the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in \mathbf{R},\, t > 0, \\ u(x,0) = g(x), & x \in \mathbf{R}, \\ u_t(x,0) = h(x), & x \in \mathbf{R}. \end{cases}$$

(A relatively brief explanation will suffice; you do not have to write all the details of the derivation.)

(b) Apply the formula from part (a) to find the solution formula for the wave equation defined on the half-line $\mathbf{R}^+ = \{x \in \mathbf{R} : \ x > 0\}$:

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in \mathbf{R}^+,\, t > 0, \\ u(x,0) = g(x), & x \in \mathbf{R}^+, \\ u_t(x,0) = h(x), & x \in \mathbf{R}^+, \\ u(0,t) = 0, & t > 0. \end{cases}$$