Ph.D. Comprehensive Exam–PDE & Fourier Analysis

January 2005

Part I: Fourier Analysis

1. Let f be the function on $[-\pi, \pi]$ defined by $f(x) = 0$ for $-\pi \le x < 0$ and $f(x) = 1$ for $0 \le x < \pi$.

   (a) Find the (trigonometric) Fourier series of f.

   (b) At what points of the interval $[-\pi, \pi]$ does the Fourier series of f converge to f? To what (if anything) does the series converge at the other points of the interval $[-\pi, \pi]$? Be sure to explain your answer.

   (c) Let g be the function on $[-\pi, \pi]$ defined by $g(x) = x^6$. Does the (trigonometric) Fourier series of g converge uniformly to g on the interval $[-\pi, \pi]$? Note: You do not need to find the Fourier series of g, but you should be sure to explain your answer.

2. All functions considered in this problem are real-valued and square-integrable on a fixed interval $[a, b]$. For such a function f we define $||f||$ as the nonnegative square root of $\int_a^b f^2(x)\, dx$. Let $\{\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots\}$ be an orthonormal system on $[a, b]$. Fix a function f and define $c_n = \int_a^b f(x)\varphi_n(x)\, dx$ for each n.

   (a) Show that for each N

   $$||f - \sum_{n=0}^{N} c_n\varphi_n||^2 = ||f||^2 - \sum_{n=0}^{N} c_n^2.$$

   (b) State the Bessel inequality and deduce it from part (a).

3. (a) Suppose that both f and g have period 2π and are absolutely integrable on $[-\pi, \pi]$. Assume that the (trigonometric) Fourier series for f converges uniformly to f on the interval $[-1, 1]$. If $f = g$ on the interval $[-2, 2]$, what can you say about the Fourier series for g?

   (b) Suppose that f has period 2π and is absolutely integrable on $[-\pi, \pi]$. Assume that f is continuous at 0. Does it follow that the (trigonometric) Fourier series for f converges at 0? Assume now that this series does converge at 0. To what must the series converge at 0, and why?

Part II: Partial Differential Equations

1. Prove the mean-value formula for Lapace's equation: if $u \in C^2(U)$ is harmonic, then for each ball $B(x,r) \subset U \subset \mathbb{R}^n$,

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y),$$

where $\partial B(x,r)$ denotes the surface of the ball centered at x with radius r, and $|\partial B(x,r)|$ denotes the surface area.

2. Derive d'Alembert's formula, namely, find the solution of the one-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & x \in \mathbb{R}. \end{cases}$$

3. Use the method of Fourier transforms to find the solution of the following problem

$$\begin{cases} u_t - \Delta u + c\, u = 0 & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}^n, \end{cases}$$

where c is a constant and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. You can use the fact that the inverse Fourier transform of

$$F(y) = e^{-t|y|^2}, \quad y \in \mathbb{R}^n$$

is given by

$$F^\vee(x) = \frac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

Recall that the inverse Fourier transform $F^\vee$ is given by

$$F^\vee(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} F(y)\, dy.$$