PhD COMPREHENSIVE EXAM— ODE & PDE — June 2017

Note: Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Ordinary Differential Equations

1. Let B denote the Banach space

$$B = \{f \in C([0,1]) : \|f\| \equiv \sup_{x\in[0,1]} |f(x)|\, e^{-x^2} < \infty\}$$

   (a) Define the map on B by

$$Tf(x) = x^3 + \int_0^x t^3 f(t)\, dt.$$

   Show that $T : B \to B$ is a contraction.

   (b) Construct the sequence $y_n(x)$ by

$$y_0(x) = 0, \quad y_n(x) = Ty_{n-1}(x), \; n = 1, 2, 3, \cdots .$$

   Does this sequence $y_n(x)$ converge as $n \to \infty$? If yes, find the limit. If no, provide your reason.

2. Let $I = [a, b]$ with $a < b$ denote an interval of the real line. Let α, β and u be continuous functions defined on I. Assume β is non-negative. Show that, if u satisfies the integral inequality

$$u(t) \le \alpha(t) + \int_a^t \beta(s)\, u(s)\, ds, \qquad \forall t \in I,$$

   then u satisfies

$$u(t) \le \alpha(t) + \int_a^t \alpha(s)\, \beta(s)\, \exp\left(\int_s^t \beta(r)\, dr\right) ds, \qquad t \in I.$$

3. (a) State the definition of stability and asymptotic stability. State Lyapunov's theorem on stability of equilibria using Lyapunov function.

   (b) Use an analysis based on Lyapunov function to determine the stability and asymptotic stability of the equilibrium at the origin of the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -Dy - \frac{g}{l} \sin x, \end{cases}$$

   where $D > 0$, $g > 0$ and $l > 0$ are constants.

Part II: Partial Differential Equations

1. Assume that $u \in C^2(\mathbb{R}^2)$ solves $\Delta u = 0$ for all $x \in \mathbb{R}^2$.

   (a) Show that for any bounded domain U with $\partial U \in C^1$, and for any $x_0 \in U$, we have

   $$u(x_0) = \frac{1}{2\pi} \int_{\partial U} \left(u(x) \frac{\partial \ln |x - x_0|}{\partial \nu} - \ln |x - x_0| \frac{\partial u}{\partial \nu}(x) \right) dS(x),$$

   where ν denotes the outer unit normal vector on ∂U.

   (b) Deduce from part (a) that

   $$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} u(\cos\theta, \sin\theta)\, d\theta.$$

2. Consider a function $u(r,t)$ of two variables r and t which solves the equation

   $$u_{tt} = u_{rr} + \frac{2}{r} u_r. \tag{1}$$

   (a) Find a general solution of this equation.

   (b) Solve the initial-value problem for (1) with $u(r,0) = g(r)$ and $u_t(r,0) = h(r)$.

3. Consider the conservation law

   $$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{2}$$

   Compute explicitly the entropy solution(s) of (2) if

   $$g(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \ge 1. \end{cases}$$

   Provide the details of your computation. Draw a graph on the upper half xt-plane to document your answer.