PhD COMPREHENSIVE EXAM— ODE & PDE — August, 2015

Note: Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Ordinary Differential Equations

1. Let $\xi, \eta \in \mathbf{R}$ and $a > 0$. Consider the initial value problem:

$$y'(x) = f(x, y), \quad \text{for} \quad \xi \leq x \leq \xi + a, \ y(\xi) = \eta,$$

where f is continuous in $[\xi, \xi + a] \times \mathbf{R}$ and there exists a constant $L > 0$ such that

$$|f(x, y) - f(x, z)| \leq L|y - z|, \ \forall (x, y), (x, z) \in [\xi, \xi + a] \times \mathbf{R}.$$

State clearly a Fixed Point Theorem for contractive mappings and use it to prove that the above problem has a unique solution $y \in C^1[\xi, \xi + a]$.

2. Let $A(t)$ be a n by n real matrix function of t and $b(t)$ a n-dimensional real vector function of t. Suppose $A(t)$ and $b(t)$ are continuous for $t \in [\xi, \xi + a]$ and there exist positive constants M, δ, γ such that

$$\|A(t)\| \leq M, \ \|b(t)\| \leq \delta, \quad \text{for } t \in [\xi, \xi + a], \ \|\eta\| \leq \gamma,$$

where $\|b(t)\|$ is a vector norm of $b(t)$ and $\|A(t)\|$ is a corresponding compatible matrix norm of $A(t)$. Prove that the solution $y(t)$ of the initial value problem

$$y'(t) = A(t)y + b(t), \quad \text{for } t \in [\xi, \xi + a], \ y(\xi) = \eta$$

exists for $t \in [\xi, \xi + a]$ and is unique. Moreover,

$$\|y(t)\| \leq \gamma e^{M(t-\xi)} + \frac{\delta}{M}(e^{M(t-\xi)} - 1).$$

3. Given $p \in C^1[a, b]$, $q \in C[a, b]$, and $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbf{R}^2$ both not $(0, 0)$, define the operators L, R_1 and R_2 as following:

$$(Lu)(x) := (p(x)u'(x))' + q(x)u(x), \quad \text{for } x \in [a, b] \ (a < b),$$

$$R_1 u := \alpha_1 u(a) + \alpha_2 u'(a), \ R_2 u := \beta_1 u(b) + \beta_2 u'(b).$$

Assume $u, v \in C^2[a, b]$ with $R_i u = R_i v = 0$ for $i = 1, 2$. Prove that

$$\int_a^b [v(x)Lu(x) - u(x)Lv(x)]dx = 0.$$

Part II: Partial Differential Equations

1. Let $x \in \mathbf{R}^2$ and $\Phi(x) = -\frac{1}{2\pi} \ln |x|$. Assume that f is twice continuously differentiable with compact support. Show that, for any $x \in \mathbf{R}^2$,

$$\lim_{\epsilon \to 0} \int_{\mathbf{R}^2 \setminus B(0,\epsilon)} \Phi(y)\, \Delta_y f(x-y)\, dy = -f(x).$$

2. Consider the wave equation with damping

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0, & x \in \mathbf{R},\ t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & x \in \mathbf{R}. \end{cases}$$

Assume that f and g are smooth and g has compact support. Let

$$E(t) = \int_{\mathbb{R}} (u_t^2 + u_x^2) dx.$$

Prove that $E(t)$ is a non increasing function of t.

3. Consider the conservation law

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x & = & 0 & \text{in } \mathbf{R} \times (0,\infty) \\ u & = & g & \text{on } \mathbf{R} \times \{t = 0\}. \end{cases} \tag{1}$$

(a) Define an integral solution of (1).

(b) Compute explicitly the unique entropy solution of (1) if

$$g(x) = \begin{cases} 1 & \text{if } x \le 0 \\ \frac{1}{2} & \text{if } 0 < x < 1 \\ 0 & \text{if } x \ge 1. \end{cases}$$

Provide the details of your computation.