PhD COMPREHENSIVE EXAM— ODE & PDE — June, 2015

Note: Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Ordinary Differential Equations

1. Consider the following initial value problem:

$$y'(t) = y^2 \cos t - e^t y, \quad \text{for } t \geq 0; \; y(0) = y_0 \in \mathbb{R}.$$

Are the following claims true or not? Justify your answers with rigorous proofs.

(a) For any y_0, there is a unique solution for $t \geq 0$ in a neighborhood of 0.

(b) For all y_0, the solutions are well defined for $t \in [0, +\infty)$.

2. Consider the following homogeneous boundary value problem:

$$Lu := (p(x)u')' + q(x)u = 0 \;\text{ for }\; x \in [0,1], \;\text{ and }\; u(0) = u(1) = 0,$$

where $p \in C^1[0,1]$, $q \in C[0,1]$ are real valued functions and $p(x) > 0$ for $x \in [0,1]$. Suppose the problem has only the trivial solution.

(a) Give the definition of Green's funtion $\Gamma = \Gamma(x, \xi)$ for the above problem.

(b) Prove that the Green's function is unique and

$$\Gamma(x, \xi) = \Gamma(\xi, x), \;\text{ for }\; (x, \xi) \in [0,1]^2.$$

3. For the following nonlinear system:

$$\frac{dx}{dt} = (1-x)y + x^2 \sin x, \; \frac{dy}{dt} = -(1-x)x + y^2 \sin y, \; (x, y \in \mathbb{R}, t \geq 0),$$

is the zero solution stable or not in the sense of Lyapunov? Prove your claim.

Part II: Partial Differential Equations

4. Let $\Phi(x) = \frac{1}{4\pi|x|}$, defined for non-zero vector $x \in \mathbb{R}^3$, denote the fundamental solution of the three-dimensional Laplace equation.

(a) Verify that $\Delta\Phi(y) = 0$ for any $y \neq 0$.

(b) Assume $f = f(x)$ with $x \in \mathbb{R}^3$ is twice continuously differentiable with compact support. Show that

$$\lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} \Phi(y) \Delta_y f(x-y)\, dy = -f(x).$$

5. Let n be a positive integer. Assume that $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, that is, g is continuous and essentially bounded in $\mathbb{R}^n$. Define

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy$$

for $x \in \mathbb{R}^n$ and $t > 0$, where $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}$ for $x \in \mathbb{R}^n$ and $t > 0$. Show that

$$\lim_{t \to 0} u(x,t) = g(x) \quad \text{for any fixed } x \in \mathbb{R}^n.$$

6. Consider the porous medium equation

$$\partial_t u - \partial_x^2(u^2) = 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty).$$

Find a solution of this equation having the form

$$u(x,t) = t^{-\alpha}\, v(xt^{-\beta}),$$

that is, to determine a function $v(y)$ and the constants α and β, such that u solves the porous medium equation. Note that you are required to find a non-constant v and positive α and β.