PhD COMPREHENSIVE EXAM— ODE & PDE — January, 2015

Note: Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Ordinary Differential Equations

1. Let $T \in \mathbb{R}$ be positive and the function $f : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ be continuous. State Ascoli-Arzela theorem and use it to prove that for a given $x_0 \in \mathbb{R}$, there is a function $x \in C^1[0,T]$ such that $x'(t) = f(t, x(t))$ for $t \in [0,T]$, and $x(0) = x_0$.

2. Find Green's function $\Gamma = \Gamma(x, \xi)$ for the boundary value problem

$$u''(t) + u(t) = 0 \ \text{ for } \ t \in [0,1], \ \text{ and } \ u(0) = u(1) = 0,$$

so that the solution(s) of the following nonlinear boundary value problem

$$u''(t) + u(t) - \sin u(t) = 0 \ \text{ for } \ t \in [0,1], \ \text{ and } \ u(0) = u(1) = 0$$

can be expressed as the fixed point(s) of the mapping T from C[0,1] to itself defined as

$$Tu(x) := \int_0^1 \Gamma(x, \xi) \sin(u(\xi)) \, d\xi.$$

Is T a contraction mapping in $C[0,1]$ with the standard maximum norm? Prove your claim.

3. Classify all the steady-state solutions of the following system as stable or unstable solutions in the sense of Lyapunov. Prove your claims.

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = -\sin x, \ (x, y \in \mathbb{R}, t \geq 0).$$

Part II: Partial Differential Equations

4. Let $\Phi(x) = -\frac{1}{2\pi} \log |x|$, defined for non-zero vector $x \in \mathbb{R}^2$, denote the fundamental solution of the 2D Laplace equation.

   (a) Verify that $\Delta\Phi(y) = 0$ for any $y \neq 0$.

   (b) Assume $f = f(x)$ with $x \in \mathbb{R}^2$ is twice continuously differentiable with compact support. Show that

$$\lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} \Phi(y) \Delta_y f(x - y) \, dy = -f(x).$$

5. Compute explicitly the unique entropy solution of

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x &= 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\ u &= g \quad \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

for

$$g(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ 2 & \text{if } \ 0 \le x \le 1 \\ 1 & \text{if } \ x > 1. \end{cases}$$

Provide the details of your computation.

6. Let $b \in \mathbb{R}^n$ be a given vector and $c > 0$ be a given number. Assume that $g \in C(\mathbb{R}^n)$ has compact support. Assume that $f = f(x, t)$ satisfies $f \in C_1^2(\mathbb{R}^n \times [0, \infty))$ and has compact support. Find an explicit solution formula of the following initial-value problem by the method of Fourier transform

$$\begin{cases} u_t + b \cdot \nabla u - c^2 \Delta u &= f \quad \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

(Hint: $C_1^2(\mathbb{R}^n \times [0, \infty))$ denotes the set of functions that are twice continuously differentiable in $x \in \mathbb{R}^n$ and continuously differentiable in $t \in (0, \infty)$. The Fourier transform $\widehat{u}$ of a function $u \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \, u(x) \, dx$$

and the inverse Fourier transform $\check{u}$ of $u \in L^1(\mathbb{R}^n)$ is given by

$$\check{u}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, u(\xi) \, d\xi.$$

In addition, you can use without proof the fact that the inverse Fourier transform of $e^{-t|\xi|^2}$ is given by $\frac{1}{(2t)^{n/2}} e^{-|x|^2/4t}$.)