PhD COMPREHENSIVE EXAM— ODE & PDE — January 2012

Note: Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Ordinary Differential Equations

1. Consider the initial-value problem

$$\begin{cases} \frac{dy}{dx} = (1 + y(x))\, f(y(x)), \\ \cancel{x}(0) = 2. \end{cases}$$

y

   (a) If $f(y)$ is locally Lipschitz in $\mathbf{R}$, what can you conclude about the solvability of this problem? Is the solution unique? Please justify your answer.

   (b) Assume $f(y) = \sqrt[3]{1+y}$. Does the solution exist for all $x \in \mathbf{R}$? Please justify your answer.

2. Let $a > 0$ and $D = \{(x, y) : |x - x_0| < a,\ y \in \mathbf{R}\}$. Assume that $f \in C(D)$ and satisfies, for $L > 0$,

$$|f(x, y_1) - f(x, y_2)| \le L\,|y_1 - y_2| \quad \text{for any } (x, y_1),\ (x, y_2) \in D.$$

   Let y be a solution of the equation $y'(x) = f(x, y(x))$ with $y(x_0) = y_0$. Suppose $z = z(x)$ satisfies $|z(x_0) - y_0| \le \gamma$ and $|z'(x) - f(x, z(x))| \le \delta$ for any x satisfying $|x - x_0| < a$, where $\gamma > 0$ and $\delta > 0$ are constants. Show that, for any x satisfying $|x - x_0| < a$,

$$|y(x) - z(x)| \le \gamma\, e^{L|x-x_0|} + \frac{\delta}{L}(e^{L|x-x_0|} - 1).$$

3. Let $J = [0, 1]$ and let $g(x)$ be continuous in J. Consider the boundary value problem

$$\begin{aligned} &((1 + x^2)u'(x))' = g(x), \quad x \in J, \\ &u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{1}$$

   (a) Are any two solutions to (1) the same? Please justify your answer.

   (b) Construct a function $G(x, y)$ defined in $J \times J$ such that u can be represented as

$$u(x) = \int_0^1 G(x, y)\, g(y)\, dy.$$

Part II: Partial Differential Equations

Notations: $\Delta u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ for function $u = u(x)$ or $u = u(x,t)$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$; $|x|$ is the Euclidean norm of x in $\mathbb{R}^n$.

4. Supose U is an open subset of $\mathbb{R}^n$ and $u \in C^2(U)$. Prove that u is harmonic in U, that is $\Delta u = 0$ in U, if and only if $u(x)$ is the mean value of u over the boundary of $B(x,r)$ for any $x \in U$ and closed ball $B(x,r) \subset U$ centered at x with Euclidean radius r.

5. Use Fourier transform and inverse Fourier transform to derive the following formula of u

$$u(x,t) = \frac{1}{(4\pi)^{n/2}} \int_0^t \frac{1}{(t-s)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s) dy ds, \quad x \in \mathbb{R}^n,\ t > 0,$$

as a solution of the initial value problem:

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $f \in C^2(\mathbb{R}^n \times [0, \infty))$ is compactly supported. Give the details of your derivation. You may use (and do *not* need to prove) the equality

$$\int_{-\infty}^{\infty} e^{iax - bx^2} dx = (\pi/b)^{1/2} e^{-a^2/4b},$$

where $a, b, x \in \mathbb{R}$, $b > 0$ and i is the imaginary unit.

6. Use the method of characteristics to find in explicit form a solution $u = u(x,y)$ of the following equation

$$x u_y(x,y) - y u_x(x,y) = u, \quad x > 0, y > 0,$$

such that $u(x,0) = x$ for $x > 0$.