PhD COMPREHENSIVE EXAM— ODE & PDE — August 2011

Note: Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Ordinary Differential Equations

1. Consider the initial-value problem (IVP)

$$\begin{cases} \frac{dy}{dx} = (e + y(x))\, f(y(x)), \\ y(0) = 5. \end{cases} \tag{1}$$

   (a) If $f(y)$ is locally lipschitz in $\mathbf{R}$, what can you conclude about the solvability of this problem? Is the solution unique? Please justify your answer.

   (b) Assume $f(y) = (\ln(e + y))^2$. Does the solution exist for all $x \in \mathbf{R}$? Please justify your answer.

2. Let B denote the Banach space

$$B = \{f \in C([0,1]) : \|f\| \equiv \sup_{x\in[0,1]} |f(x)|\, e^{-x^2} < \infty\}$$

   (a) Define the map on B by

$$Tf(x) = x^3 + \int_0^x t^3 f(t)\, dt.$$

   Show that $T : B \to B$ is a contraction.

   (b) Construct the sequence $y_n(x)$ by

$$y_0(x) = 0, \quad y_n(x) = Ty_{n-1}(x), \; n = 1, 2, 3, \cdots .$$

   Does this sequence $y_n(x)$ converge uniformly as $n \to \infty$? If yes, find the limit. If no, provide your reason.

3. Let $J = [0, 1]$ and let $g(x)$ be continuous in J. Consider the boundary value problem

$$\begin{aligned} &u''(x) - u(x) = g(x), \quad x \in J, \\ &u(0) = 0, \quad u'(1) = 0. \end{aligned} \tag{2}$$

   (a) Show that any two solutions of (2) must be the same.

   (b) Let $g(x) = e^{4x}$. Construct a function $G(x, y)$ defined in $J \times J$ such that u can be represented as

$$u(x) = \int_0^\pi G(x, y)\, g(y)\, dy$$

   and compute u explicitly.

Part II: Partial Differential Equations

4. Assume Ω is a bounded open subset of $\mathbb{R}^2$ with boundary $\partial\Omega$ being C^1, $u \in C^2(\bar{\Omega})$ and $u(x) = 0$ for $x \in \partial\Omega$. Let $\frac{\partial u}{\partial \nu}$ be the outer normal derivative of u on $\partial\Omega$. Prove with details that, for $x \in \Omega$,

$$u(x) = \frac{1}{2\pi}\int_{\Omega} \Delta u(y) \log|x-y| dy - \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial u}{\partial \nu}(y) \log|x-y| dS(y),$$

and both of the above integrals are finite. If you use a version of divergence theorem or Gauss-Green theorem, state it clearly.

5. Consider the initial value problem for the wave equation:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g, \; u_t = f & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose that g, f are smooth and compactly supported in $\mathbb{R}$. Use the method of Fourier transform to derive *d'Alembert's* formula of the solution of the above problem. You are not required to justify the result is indeed a solution. However, provide details for evaluation of the involved inverse Fourier transform.

6. Consider the following initial value problem

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

where

$$g(x) = \begin{cases} -1 & \text{if } x \in (-\infty, 0), \\ 1 & \text{if } x \in (0, 1), \\ 0 & \text{if } x \in (1, \infty). \end{cases}$$

   (a) How many entropy solutions of the above problem are there? Cite and state a theorem to answer this question.

   (b) Compute explicitly the entropy solution(s) with justification. Draw a graph on the upper half xt-plane to document your answer.