PhD COMPREHENSIVE EXAM— ODE & PDE — May/June 2011

Note: Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Ordinary Differential Equations

1. Consider the following system of equations:

$$\begin{cases} \frac{dx_1}{dt} = x_1(2 - x_2), \\ \frac{dx_2}{dt} = x_2(-3 + 4x_1). \end{cases} \tag{1}$$

   (a) Find a non-constant function $F = F(x_1, x_2)$ such that $F(x_1(t), x_2(t))$ is a constant for any solution (x_1, x_2) of (1).

   (b) Given an initial datum $(x_1(0), x_2(0)) = (\xi, \eta)$. Is the solution of (1) unique? Is the solution periodic? Justify your answer.

2. Let D denote the domain $D = \{(x, y) : |x - x_0| < a,\ y \in \mathbf{R}\}$. Assume that $f \in C(D)$ and satisfies

$$|f(x, y_1) - f(x, y_2)| \le \phi(x)\, |y_1 - y_2| \quad \text{for any } (x, y_1),\ (x, y_2) \in D,$$

   where $\phi(x)$ is a nonnegative continuous function on the open interval $(x_0 - a, x_0 + a)$. Prove the uniqueness of the solutions to the equation, for any $|x - x_0| < a$,

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))\, dt.$$

3. Let $J = [0, \pi]$ and let $g(x)$ be continuous in J. Consider the boundary value problem

$$\begin{aligned} &u''(x) + u(x) = g(x), \quad x \in J, \\ &au(0) + bu'(0) = 0, \quad cu(\pi) + du'(\pi) = 0. \end{aligned} \tag{2}$$

   (a) Are any two solutions to (2) always the same? If yes, give a proof and if no, provide an example.

   (b) Let $g(x) = \sin x$. Let $a = c = d = 1$ and $b = 0$. Construct a function $G(x, y)$ defined in $J \times J$ such that u can be represented as

$$u(x) = \int_0^{\pi} G(x, y)\, g(y)\, dy$$

   and compute u explicitly.

Part II: Partial Differential Equations

4. Supose $f \in C^2(\mathbb{R}^2)$ is compactly supported. Define, for $x \in \mathbb{R}^2$,

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log |x-y| dy.$$

Give a direct and detailed proof that $u \in C^2(\mathbb{R}^2)$ and $\Delta u = f$ in $\mathbb{R}^2$. If you use a version of divergence theorem or Gauss-Green theorem, state it clearly.

5. Consider the initial value problem for the heat equation:

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Suppose that $g \in C(\mathbb{R}^n)$ is compactly supported. Use the method of Fourier transform to derive the formula of a solution of the above problem. You are not required to provide details for evaluation of the involved inverse Fourier transform. However, justify that your formula indeed gives a solution of the heat equation. In what sense is the boundary condition satisfied? Justify your answer with a detailed proof.

6. Use the method of characteristics to find a solution $u = u(x, y)$ of the following equation

$$x u_x(x, y) + y u_y(x, y) = \sec(u(x, y))$$

such that $u(x, x^3) = 0$.