PhD COMPREHENSIVE EXAM— ODE & PDE — January 2011

Note: Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Ordinary Differential Equations

1. Consider the following initial value problem:

$$x''(t) + x'(t) + \sin(x(t)) = 0, \text{ for } t \in \mathbf{R}; \quad x(0) = x_0, \; x'(0) = x_1.$$

   (a) Prove that for any $(x_0, x_1) \in \mathbf{R}^2$, there exists a unique solution $x = x(t)$ of the above problem for <u>all</u> $t \in (-\infty, \infty)$.

   (b) Consider the solution $x = x(t)$ for $(x_0, x_1) = (\pi, 0)$. Is this solution a stable, asymptotically stable or unstable solution? Justify your answer.

2. Let D denote the annulus $D = \left\{x = (x_1, x_2) \in \mathbf{R}^2 : \frac{1}{2} < |x| < 3\right\}$, where $|x| = \sqrt{x_1^2 + x_2^2}$. Let $\mathbf{u}$ be the vector field given by $\mathbf{u} = (u_1(x_1, x_2), u_2(x_1, x_2)) = (-\partial_{x_2} \ln|x|, \partial_{x_1} \ln|x|)$.

   (a) Show that the solution of the system of ordinary differential equations

$$\frac{dx_1}{dt} = u_1(x_1, x_2), \quad \frac{dx_2}{dt} = u_2(x_1, x_2), \quad x_1(0) = 2, \quad x_2(0) = 0$$

   is periodic in t.

   (b) Show that $\nabla \times \mathbf{u} = \mathbf{0}$ in D and that there is not a differentiable function p such that $\mathbf{u} = \nabla p$ for every $(x_1, x_2) \in D$.

3. Consider the boundary-value problem

$$u''(x) = g(x)\, e^{u(x)} \quad \text{in} \quad [0, 1], \qquad u(0) = u(1) = 0. \tag{1}$$

   where $g \in C[0, 1]$ is given. Show that there is one and only one solution if $0 \le g(x) \le L < 8$ for $x \in [0, 1]$ by following the steps below.

   (a) Construct a Green's function $\Gamma(x, \xi)$ for

$$u''(x) = 0 \quad \text{in} \quad [0, 1], \qquad u(0) = u(1) = 0.$$

   (b) Write (1) as the integral

$$u(x) = (Tu)(x) \equiv \int_0^1 \Gamma(x, \xi)\, f(\xi, u(\xi))\, d\xi \tag{2}$$

   where $f(\xi, u(\xi)) = g(\xi)\, e^{u(\xi)}$. Apply a fixed point argument to show that the integral equation (2) has a unique solution.

PhD COMPREHENSIVE EXAM— ODE & PDE — January 2011

Part II: Partial Differential Equations

1. Let $U \subset \mathbb{R}^n$ be open, bounded, and connected. We say $u \in C^2(U) \cap C(\overline{U})$ is harmonic if $\Delta u = 0$ in U.

   (a) Suppose u is harmonic and $u \geq 0$ in U. Use the Mean Value Formula to prove that either $u > 0$ in U or $u \equiv 0$ in U.

   (b) Let u, v be harmonic and $u \geq v$ on ∂U. Assume u is not identically equal to v on ∂U, that is, there exists $\boldsymbol{x} \in \partial U$ such that $u(\boldsymbol{x}) \neq v(\boldsymbol{x})$. Prove that $u > v$ in U.

2. (a) (Duhamel's principle) Consider the nonhomogeneous wave equation

$$\text{(I)} \quad \begin{cases} u_{tt} - \Delta u = f(\boldsymbol{x}, t) & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = 0, \ u_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

   Let $v(\boldsymbol{x}, t; s)$ be the solution of

$$\begin{cases} v_{tt}(\boldsymbol{x}, t; s) - \Delta v(\boldsymbol{x}, t; s) = 0 & \text{in } \mathbb{R}^n \times (s, \infty), \\ v(\boldsymbol{x}, t; s) = 0, \ v_t(\boldsymbol{x}, t; s) = f(\boldsymbol{x}, s) & \text{on } \mathbb{R}^n \times \{t = s\}. \end{cases}$$

   Show that u defined as following is a solution of problem (I):

$$u(\boldsymbol{x}, t) := \int_0^t v(\boldsymbol{x}, t; s)\, ds.$$

   (b) Use the above result to solve the 1-D wave equation

$$\begin{cases} u_{tt} - u_{xx} = 2t + 4x & \text{in } \mathbb{R} \times (0, \infty), \\ u = 0, \ u_t = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

3. Use the method of characteristics to solve for $u(x, t)$:

$$\begin{cases} x u_t + u^2 u_x = 0 & \text{in } \mathbb{R} \times \mathbb{R}, \\ u(x, 0) = x & \text{for } x \in \mathbb{R}. \end{cases}$$