PhD COMPREHENSIVE EXAM—ODE & PDE—January 2008

Part I: Ordinary Differential Equations

1. Let p, q, r be continuous real-valued functions on $\mathbf{R}$ with $p > 0$. Prove that the equation
$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0$$
is equivalent to
$$(a(t)x'(t))' + b(t)x(t) = 0,$$
where a is continuously differentiable and b is continuous.

2. Consider the system $r' = r(r-1)(r-2)^2(3-r), \theta' = 1$ in polar coordinates. Find all periodic orbits and classify as to stability.

3. Prove that every solution $x(t)$ of the differential equation
$$\frac{dx}{dt} = x^2 - x^5, x(0) > 0,$$
satisfies $\lim_{t\to\infty} x(t) = 1$.

Part II: Partial Differential Equations

1. For the equation $u_x^2 + u_y^2 = u^2$, find (a) the characteristic strips; (b) the integral surfaces through the line $x = s, y = 0, z = 1$.

2. For the $n = 3$ dimensional case,

   (a) Prove that the spherically symmetric solution of the hyperbolic equation $u_{tt} - c^2 \Delta u = 0$ about the origin has the form

   $$u(x, y, z) = \frac{F(r + ct) + G(r - ct)}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

   (b) Prove that the solution in part (a) with initial data $u(x, 0) = 0, u_t(x, 0) = g(r)$ (with $g(r)$ an even function of r) is given by

   $$u(x, y, z) = \frac{1}{2cr} \int_{r-ct}^{r+ct} \rho g(\rho) d\rho.$$

   (c) Find an explicit solution in part (b) for the function g defined by $g(r) = 1$ if $0 < r < a$ and $g(r) = 0$ for $r > a$.

3. Let $n = 2$ and let Ω be the half-plane with $x_2 > 0$. Let $\xi^* = (\xi_1, -\xi_2)$ be a reflection of $\xi = (\xi_1, \xi_2)$.

   (i) Prove that $G(x, \xi) = K(x, \xi) - K(x, \xi^*)$ is a Green function for Ω.

   (ii) Prove that the corresponding Poisson formula for the boundary problem $u|_{\partial\Omega} = f(x_1)$ is given by

   $$u(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\xi_2 f(x_1)}{(x_1 - \xi_1)^2 + \xi_2^2} dx_1.$$

   (iii) Prove that the maximum principle is satisfied by the solution in the above part (ii).