PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & ODE—June 2010

Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Fourier Analysis

1. (a) State the Riemann-Lebesgue lemma.

   (b) Let f be continuous on the real line and 2π-periodic. Assume that f is differentiable at 0. Use the Riemann-Lebesgue lemma to show that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} (f(x) - f(0))D_n(x)\, dx = 0.$$

   Here $D_n(x) = \dfrac{\sin((n+1/2)x)}{\sin(x/2)}$ for $x \in [-\pi, \pi]$ and $x \neq 0$.

2. Be sure to give a brief reason for your answer in each part.

   (a) Is the trigonometric series $\sum_{n=1}^{\infty} [1 + (-1)^n] \sin nx$ the Fourier series of an integrable function on $[-\pi, \pi]$ (that is, a function in $L^1([-\pi, \pi])$)?

   (b) Is the trigonometric series $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/4}} \cos nx$ the Fourier series of a square-integrable function on $[-\pi, \pi]$ (that is, a function in $L^2([-\pi, \pi])$)?

   (c) At what points of the interval $[-\pi, \pi]$ does the trigonometric series $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/4}} \cos nx$ converge?

3. This problem concerns the Fourier (integral) transform of functions integrable on the real line $\mathbf{R}$ (that is, functions belonging to $L^1(\mathbf{R})$).

   (a) State one version of the Fourier inversion theorem.

   (b) Show that if f is even and the Fourier inversion theorem applies to f at x then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos(\lambda x) \left[\int_0^{\infty} f(t) \cos(\lambda t)\, dt \right] d\lambda.$$

   (c) Use part (b) to prove

$$\int_0^{\infty} \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2}.$$

   Suggestion: You may assume that the Fourier inversion theorem applies to f at 0, where

$$f(t) = \begin{cases} 1 & \text{for} \quad -1 < t < 1 \\ 0 & \text{for} \quad |t| > 1 \\ \frac{1}{2} & \text{for} \quad |t| = 1. \end{cases}$$

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Part II: Ordinary Differential Equations

1. Let Ω be the domain $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1, -2 \leq x_2 \leq 2\}$. Consider the following system of equations:

$$\begin{cases} \frac{dx_1}{dt} = g_1(x_1, x_2, t), & (x_1, x_2) \in \Omega,\ t \in \mathbb{R}, \\ \frac{dx_2}{dt} = g_2(x_1, x_2, t), & (x_1, x_2) \in \Omega,\ t \in \mathbb{R}, \end{cases} \tag{1}$$

where $g_1(x_1, x_2, t) = x_2$ and $g_2(x_1, x_2, t) = -\sin(x_1) - x_1|x_2|^2 + \cos t$.

   (a) Are g_1 and g_2 Lipschitz in the domain for $(x_1, x_2) \in \Omega$ and $t \in \mathbb{R}$? Rigorously verify your answer.

   (b) For the initial data $x_1(0) = 0$ and $x_2(0) = 0$, does this system have a solution at least for $|t| \leq \delta$ with some $\delta > 0$? Is there only one solution? Justify your answer.

   (c) If the answer to the first question in (b) is yes, find the maximal $\delta > 0$ so that the solution exists and remains in Ω for $|t| \leq \delta$.

2. Consider the initial-value problem

$$\begin{cases} \frac{dx}{dt} = (1 + x(t))\,(\ln(2 + x(t)))^{\beta}, \\ x(0) = e. \end{cases} \tag{2}$$

Determine all values of β for which the solution of (2) has a finite-time singularity, namely, there exists a $T \in (0, \infty)$,

$$\lim_{t \to T} x(t) = \infty.$$

Justify your answer.

3. Let

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Find the explicit solution to the initial-value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

   (b) Determine the stable and unstable subspaces of this linear system.