PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & ODE—January 2010

Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Four problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Fourier Analysis

1. Let f be the function on $[-\pi, \pi]$ defined by $f(x) = 0$ if $-\pi \leq x < 0$ and $f(x) = x$ if $0 \leq x \leq \pi$.

   (a) Find the (trigonometric) Fourier series of f.

   (b) At what points of the interval $[-\pi, \pi]$ does the Fourier series converge to f? To what (if anything) does the series converge at the other points of the interval $[-\pi, \pi]$? Be sure to explain your answers.

   (c) Does the Fourier series of f converge uniformly on $[0, 1]$? Be sure to explain your answer.

2. Be sure to give a brief reason for your answer in each part.

   (a) At what points of the interval $[-\pi, \pi]$ does the trigonometric series $\sum_{n=1}^{\infty} \frac{1}{n} \sin nx$ converge?

   (b) Let f be the function defined as the sum of the series in part (a). At what points of the interval $[-\pi, \pi]$ is f continuous? Give as complete an answer as you can.

   (c) Let g be the function defined as the sum of the trigonometric series $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$. At what points of the interval $[-\pi, \pi]$ is g continuous?

3. All functions considered in this problem are real-valued and square-integrable on a fixed interval $[a, b]$. For such a function f we define $||f||$ as the nonnegative square root of $\int_a^b f(x)^2\, dx$. Let $\{\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots\}$ be an orthonormal system on $[a, b]$. In this problem you may assume the following fact: If $N \geq 0$ and $\gamma_0, \gamma_1, \ldots, \gamma_N$ are real numbers, then

$$\left|\left|\sum_{n=0}^{N} \gamma_n \varphi_n\right|\right|^2 = \sum_{n=0}^{N} \gamma_n^2.$$

   (a) Let f have Fourier coefficients $c_0, c_1, \ldots$ with respect to the system $\{\varphi_0, \varphi_1, \ldots\}$. Show that, for all $N \geq 0$,

$$\left|\left|f - \sum_{n=0}^{N} c_n \varphi_n\right|\right|^2 = ||f||^2 - \sum_{n=0}^{N} c_n^2.$$

   (b) Give two equivalent formulations of what it means for an orthonormal system to be complete, and prove that they are equivalent.

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Part II: Ordinary Differential Equations

1. Establish the uniqueness of solutions to the following initial-value problem

$$\begin{cases} \frac{dx}{dt} = (\sin x)^2\, g(x) \, + \, e^{3t}, \\ x(0) = x_0, \end{cases}$$

where $g : \mathbf{R} \to \mathbf{R}$ is bounded and Lipschitz, namely, for $M > 0$ and $L > 0$,

$$|g(x)| \leq M, \qquad |g(x) - g(y)| \leq \, L|x - y|$$

for any $x, y \in \mathbf{R}$.

2. Consider the initial-value problem

$$\begin{cases} \frac{dx}{dt} = x + x^{\alpha}, \\ x(0) = e^{-1}. \end{cases} \tag{1}$$

(a) Determine the values of α for which the solution of (1) exists for all time $t \geq 0$. Justify your answer.

(b) Determine the values of α for which the solution of (1) has a finite-time singularity, namely, for $0 < T < \infty$,

$$\lim_{t \to T} x(t) = \infty.$$

Justify your answer.

3. Consider the system

$$\begin{cases} \frac{dx}{dt} = x \, \cos t - y \, \sin t, \\ \frac{dy}{dt} = x \, \sin t + y \, \cos t. \end{cases} \tag{2}$$

(a) Let $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ and $\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}$ be two solutions of (2) with

$$\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} x_2(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Compute the determinant of $X(t)$, where $X(t)$ is the matrix

$$X(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix}.$$

(b) Show that every solution of (2) is periodic.