PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & ODE—August 2009

Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Five problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

## Part I: Fourier Analysis

1. In this problem we consider the series $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$ on the open interval $0 < x < 2\pi$. You may assume that this is the Fourier series of a function $f$ that is not only $2\pi$-periodic and integrable on the closed interval $0 \le x \le 2\pi$ but also continuous on the open interval.

   (a) Write $s_n$ for the $n$th partial sum of the Fourier series of $f$, and define $\sigma_n = (s_0 + s_1 + \cdots + s_{n-1})/n$. What can you conclude about the convergence of $\{\sigma_n\}$ at points of the open interval?

   (b) Does the Fourier series converge to $f$ at points of the open interval? Be sure to explain your answer.

2. Be sure to justify your answer in each part.

   (a) Let $f$ be a square integrable function on the interval $-\pi \le x \le \pi$, so $f \in L^2([-\pi, \pi])$. Assume that all the Fourier coefficients of $f$ are 0. What can you conclude about $f$?

   (b) Let $f$ be a continuous function on the interval $-\pi \le x \le \pi$. Assume that the Fourier series of $f$ converges to 0 at each point of the interval. What can you conclude about the Fourier coefficients of $f$?

   (c) Let $f$ be a function on the interval $-\pi \le x \le \pi$. Assume that there is a *trigonometric* series $\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$ that converges uniformly to $f$ on $-\pi \le x \le \pi$. Show that $f$ is integrable on $-\pi \le x \le \pi$ and that the trigonometric series is the Fourier series of $f$, that is, the coefficients $\{\alpha_n\}$ and $\{\beta_n\}$ equal the Fourier coefficients of $f$.

3. This problem concerns the Fourier (integral) transform of functions integrable on the real line $\mathbf{R}$.

   (a) Assume that both $f$ and $xf(x)$ are integrable on $\mathbf{R}$. State without proof a formula for the derivative of the Fourier transform of $f$.

   (b) For the remainder of the problem we use $\phi$ to denote the Fourier transform of the function $f(x) = e^{-x^2}$. Explain why the formula in part (a) applies, and write down what it means for $\phi$. Use this to find a differential equation relating $\phi'$ to $\phi$.

   (c) Solve the differential equation in part (b) to find a formula for $\phi$. You may assume that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

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## Part II: Ordinary Differential Equations

1. State (without proof) a local existence and uniqueness theorem for the initial value problem
$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0.$$
Does the problem
$$x' = x^{1/3}, \qquad x(0) = 0,$$
satisfy the conditions of the above theorem? Explain the reason. And give at least two different solutions to this problem for $t \geq 0$.

2. Consider the following equation in Sturm-Liouville form:
$$L[u] \triangleq -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = \lambda r(x)u,$$
where $u$ is a function of the free variable $x$, and the coefficients satisfy $p(x) > 0$, $r(x) > 0$ for $x \in [a, b]$. We assume that $p(x)$, $q(x)$, $r(x)$ are continuous on $[a, b]$ and, furthermore, that $p(x)$ has a continuous first-order derivative on $[a, b]$.

   (a) Show that the operator $L$, under the periodic boundary condition $u(a) = u(b)$ and $p(a)u'(a) = p(b)u'(b)$, is self-adjoint in the sense that $\int_a^b L[u]\, v\, dx = \int_a^b L[v]\, u\, dx$ for all $u$, $v$ that have continuous second-order derivatives.

   (b) Let $p(x) = r(x) = 1$, $q(x) = 0$, $[a, b] = [-\pi, \pi]$. Consider the periodic boundary problem defined in part (a). Prove that all non-zero eigenvalues are multiple eigenvalues, that is, to each eigenvalue there correspond more than one linearly independent eigenfunctions.

3. Assume a two-variable function $f(x, y)$ has continuous second-order derivatives and $\lim_{x^2+y^2\to\infty} f(x, y) = +\infty$. Consider the following gradient system:
$$\begin{cases} \frac{dx}{dt} = -\frac{\partial f}{\partial x}(x(t), y(t)) \\ \frac{dy}{dt} = -\frac{\partial f}{\partial y}(x(t), y(t)), \end{cases}$$
with the initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$.

   (a) Prove the above initial value problem has a unique global solution $(x(t), y(t))$ for $t \geq t_0$.

   (b) If $(\bar{x}, \bar{y})$ is an isolated local minimum of $f(x, y)$, show that it must be a stable critical point of the gradient system.